Exercise 2.2

Question 1:

Prove that $3sin^{-1}x = sin^{-1}(3x - 4x^3),\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Answer 1:

Let $sin^{-1}x = \theta$, then $x = \sin\theta$. We have,

RHS $= sin^{-1}(3x - 4x^3) = sin^{-1}(3\sin\theta - 4sin^3\theta)$

$= sin^{-1}(\sin 3\theta) = 3\theta = 3sin^{-1}x =$ LHS

Question 2:

Prove that $3cos^{-1}x = cos^{-1}(4x^3 - 3x),\ x \in \left[\frac{1}{2}, 1\right]$.

Answer 2:

Let $cos^{-1}x = \theta$, then $x = \cos\theta$. We have,

RHS $= cos^{-1}(4x^3 - 3x) = cos^{-1}(4cos^3\theta - 3cos\theta)$

$= cos^{-1}(\cos 3\theta) = 3\theta = 3cos^{-1}x =$ LHS

Question 3:

Prove that $tan^{-1}\frac{2}{11} + tan^{-1}\frac{7}{24} = tan^{-1}\frac{1}{2}$

Answer 3:

LHS $= tan^{-1}\frac{2}{11} + tan^{-1}\frac{7}{24}$

$$= tan^{-1}\left(\frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11} \times \frac{7}{24}}\right) = tan^{-1}\left(\frac{\frac{48 + 77}{11 \times 24}}{\frac{11 \times 24 - 14}{11 \times 24}}\right)$$

$$= tan^{-1}\frac{48 + 77}{264 - 14} = tan^{-1}\frac{125}{251} = tan^{-1}\frac{1}{2} = \text{RHS}$$

Question 4:

Prove that $2tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{7} = tan^{-1}\frac{31}{17}$

Answer 4:

LHS $= 2tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{7}$

$$= tan^{-1}\left[\frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2}\right] + tan^{-1}\frac{1}{7} = tan^{-1}\frac{1}{\left(\frac{3}{4}\right)} + tan^{-1}\frac{1}{7}$$

$$= tan^{-1}\frac{4}{3} + tan^{-1}\frac{1}{7} = tan^{-1}\left(\frac{\frac{4}{3} + \frac{1}{7}}{1 - \frac{4}{3} \times \frac{1}{7}}\right)$$

$$= tan^{-1}\left(\frac{\frac{28 + 3}{3 \times 7}}{\frac{3 \times 7 - 4}{3 \times 7}}\right) = tan^{-1}\frac{28 + 3}{21 - 4} = tan^{-1}\frac{31}{17} = \text{RHS}$$

Question 5:

Write the function $tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, x \neq 0$, in the simplest form.

Answer 5:

Given function $tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

Let $x = \tan\theta$

$$\therefore tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = tan^{-1}\frac{\sqrt{1+tan^2\theta}-1}{\tan\theta}$$

$$= tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= tan^{-1}\left(\frac{2sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}cos\frac{\theta}{2}}\right) = tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2} = \frac{1}{2}tan^{-1}x$$

Question 6:

Write the function $tan^{-1}\frac{1}{\sqrt{x^2-1}}, |x| > 1$, in the simplest form.

Answer 6:

Given function $tan^{-1}\frac{1}{\sqrt{x^2-1}}$

Let $x = \text{cosec}\,\theta$

$$\therefore tan^{-1}\frac{1}{\sqrt{x^2-1}} = tan^{-1}\frac{1}{\sqrt{cosec^2\theta-1}}$$

$$= tan^{-1}\frac{1}{\cot\theta} = tan^{-1}\tan\theta = \theta = cosec^{-1}x$$

$$= \frac{\pi}{2} - sec^{-1}x$$

Question 7:

Write the function $tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), x < \pi,$ in the simplest form.

Answer 7:

The given function is $tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$, Now,

$$tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right) = tan^{-1}\left(\sqrt{\frac{2sin^2\frac{x}{2}}{2cos^2\frac{x}{2}}}\right)$$

$$= tan^{-1}\left(\sqrt{tan^2\frac{x}{2}}\right) == tan^{-1}\left(tan\frac{x}{2}\right) = \frac{x}{2}$$

Question 8:

Write the function $tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right), 0<x<\pi,$ in the simplest form.

Answer 8:

The given function is $tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$

Now,

$$tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)=tan^{-1}\left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)=tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$=tan^{-1}\left(\frac{1-\tan x}{1+1.\tan x}\right)=tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}.\tan x}\right)$$

$$=tan^{-1}\left[tan\left(\frac{\pi}{4}-x\right)\right]=\frac{\pi}{4}-x$$

Question 9:

Write the function $tan^{-1}\frac{x}{\sqrt{a^2-x^2}}, |x|<a,$ in the simplest form.

Answer 9:

The given function is $tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$.

Let $x=\text{a}\sin\theta$

$$\therefore tan^{-1}\frac{x}{\sqrt{a^2-x^2}}=tan^{-1}\left(\frac{\text{a}\sin\theta}{\sqrt{a^2-a^2sin^2\theta}}\right)=tan^{-1}\left(\frac{\text{a}\sin\theta}{a\sqrt{1-sin^2\theta}}\right)$$

$$=tan^{-1}\left(\frac{\text{a}\sin\theta}{\text{a}\sin\theta}\right)=tan^{-1}(\tan\theta)=\theta=sin^{-1}\frac{x}{a}$$

Question 10:

Write the function in $tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right), a>0; \frac{-a}{\sqrt{3}}\leq x\frac{a}{\sqrt{3}},$ the simplest form.

Answer 10:

The given function is $tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$

Let $x=a\tan\theta$

$$\therefore tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)=tan^{-1}\left(\frac{3a^2.a\tan\theta-a^3tan^3\theta}{a^3-3a.a^2tan^2\theta}\right)$$

$$=tan^{-1}\left(\frac{3a^3tan\,\theta-a^3tan^3\theta}{a^3-3a^3tan^2\theta}\right)$$

$$=tan^{-1}\left(\frac{3tan\,\theta-tan^3\theta}{1-3tan^2\theta}\right)$$

$$=tan^{-1}(tan3\theta)=3\theta$$

$$=3tan^{-1}\frac{x}{a}$$

Question 11:

Find the value of $tan^{-1}\left[2cos\left(2sin^{-1}\frac{1}{2}\right)\right]$

Answer 11:

The given function is $tan^{-1}\left[2cos\left(2sin^{-1}\frac{1}{2}\right)\right]$

$\therefore tan^{-1}\left[2cos\left(2sin^{-1}\frac{1}{2}\right)\right] = tan^{-1}\left[2cos\left(2sin^{-1}\left(sin\frac{\pi}{6}\right)\right)\right]$

$= tan^{-1}\left[2cos\left(2\times\frac{\pi}{6}\right)\right] = tan^{-1}\left[2cos\left(\frac{\pi}{3}\right)\right] = tan^{-1}\left[2\times\frac{1}{2}\right]$

$= tan^{-1}[1] = \frac{\pi}{4}$

Question 12:

Find the value of $cot(tan^{-1}a + cot^{-1}a)$.

Answer 12:

The given function is $cot(tan^{-1}a + cot^{-1}a)$.

$\therefore cot(tan^{-1}a + cot^{-1}a) = cot\left(\frac{\pi}{2}\right) = 0$ [as $tan^{-1}x + cot^{-1}x = \frac{\pi}{2}$]

Question 13:

Find the value of $tan\frac{1}{2}\left[sin^{-1}\frac{2x}{1+x^2} + cos^{-1}\frac{1-y^2}{1+y^2}\right], |x| < 1, y > 0$ and $xy < 1$.

Answer 13:

The given function is $tan\frac{1}{2}\left[sin^{-1}\frac{2x}{1+x^2} + cos^{-1}\frac{1-y^2}{1+y^2}\right]$

$\therefore tan\frac{1}{2}\left[sin^{-1}\frac{2x}{1+x^2} + cos^{-1}\frac{1-y^2}{1+y^2}\right]$

$= tan\frac{1}{2}[2tan^{-1}x + 2tan^{-1}y]$ [as $2tan^{-1}x = sin^{-1}\frac{2x}{1+x^2} = cos^{-1}\frac{1-x^2}{1+x^2}$]

$= tan\frac{1}{2}[2(tan^{-1}x + tan^{-1}y)] = tan[tan^{-1}x + tan^{-1}y]$

$= tan\left[tan^{-1}\frac{x+y}{1-xy}\right] = \frac{x+y}{1-xy}$

Question 14:

If $sin\left(sin^{-1}\frac{1}{5} + cos^{-1}x\right) = 1$, then find the value of *x*.

Answer 14:

Since, $sin\left(sin^{-1}\frac{1}{5} + cos^{-1}x\right) = 1$

$\therefore \left(sin^{-1}\frac{1}{5} + cos^{-1}x\right) = sin^{-1}1$

$\Rightarrow \left(sin^{-1}\frac{1}{5} + cos^{-1}x\right) = \frac{\pi}{2}$

$\Rightarrow sin^{-1}\frac{1}{5} = sin^{-1}x$ [as $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$]

$\Rightarrow x = \frac{1}{5}$

Question 15:

If $tan^{-1}\frac{x-1}{x-2}+tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$, then find the value of x.

Answer 15:

Given that $tan^{-1}\frac{x-1}{x-2}+tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

$$\Rightarrow tan^{-1}\left(\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\times\frac{x+1}{x+2}}\right)=\frac{\pi}{4}\quad\left[as\ tan^{-1}x+tan^{-1}y=tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow \frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\times\frac{x+1}{x+2}}=tan\frac{\pi}{4}\ \Rightarrow \frac{\left[\frac{(x-1)(x+2)+(x-2)(x+1)}{(x-2)(x+2)}\right]}{\left[\frac{(x-2)(x+2)-(x-1)(x+1)}{(x-2)(x+2)}\right]}=1$$

$$\Rightarrow \frac{x^2+2x-x-2+x^2+x-2x-2}{x^2-4-(x^2-1)}=1\ \Rightarrow \frac{2x^2-4}{-3}=1$$

$$\Rightarrow 2x^2-4=-3\ \Rightarrow x^2=\frac{1}{2}\ \Rightarrow x=\pm\frac{1}{\sqrt{2}}.$$

Question 16:

Find the values of $sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

Answer 16:

Given that $sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

We know that $\sin^{-1}(\sin x)=x$ if $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, which is the principal value branch of $\sin^{-1}x$.

$$\therefore sin^{-1}\left(\sin\frac{2\pi}{3}\right)=sin^{-1}\left(\sin\left\{\pi-\frac{\pi}{3}\right\}\right)=sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

Hence, $sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\frac{\pi}{3}$

Question 17:

Find the values of $tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

Answer 17:

Given that $tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

We know that $\tan^{-1}(\tan x)=x$ if $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

$$\therefore tan^{-1}\left(\tan\frac{3\pi}{4}\right)=tan^{-1}\left(\tan\left\{\pi-\frac{\pi}{4}\right\}\right)=tan^{-1}\left(-\tan\frac{\pi}{4}\right)$$

$$=tan^{-1}\left(\tan\left\{-\frac{\pi}{4}\right\}\right)=-\frac{\pi}{4}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

Hence, $tan^{-1}\left(\tan\frac{3\pi}{4}\right)=-\frac{\pi}{4}$

Question 18:

Find the values of $tan\left(sin^{-1}\frac{3}{5}+cot^{-1}\frac{3}{2}\right)$.

Answer 18:

Given that $tan\left(sin^{-1}\frac{3}{5}+cot^{-1}\frac{3}{2}\right)$

$$\therefore tan\left(sin^{-1}\frac{3}{5}+cot^{-1}\frac{3}{2}\right)=tan\left(tan^{-1}\frac{3}{\sqrt{5^2-3^2}}+tan^{-1}\frac{2}{3}\right)$$

$$\left[as\ sin^{-1}\frac{a}{b}=tan^{-1}\frac{a}{\sqrt{b^2-a^2}}\ and\ cot^{-1}\frac{a}{b}=tan^{-1}\frac{b}{a}\right]$$

$$=tan\left(tan^{-1}\frac{3}{4}+tan^{-1}\frac{2}{3}\right)$$

$$=tan\left[tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)\right]$$

$$=tan\left[tan^{-1}\left(\frac{\frac{9+8}{4\times 3}}{\frac{4\times 3-3\times 2}{4\times 3}}\right)\right]$$

$$=tan\left(tan^{-1}\frac{17}{6}\right)=\frac{17}{6}$$

Question 19:

$cos^{-1}\left(cos\frac{7\pi}{6}\right)$ is equal to

(A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Answer 19:

Given that $cos^{-1}\left(cos\frac{7\pi}{6}\right)$

We know that $\cos^{-1}(\cos x)=x$, if $x\in[0,\pi]$, which is the principal value branch of $\cos^{-1}x$.

$$\therefore cos^{-1}\left(cos\frac{7\pi}{6}\right)$$

$$=cos^{-1}\left[cos\left(2\pi-\frac{5\pi}{6}\right)\right]$$

$$=cos^{-1}\left(cos\frac{5\pi}{6}\right)=\frac{5\pi}{6}\in[0,\pi]$$

Hence, $cos^{-1}\left(cos\frac{7\pi}{6}\right)=\frac{5\pi}{6}$

Hence, the option (B) is correct.

Question 20: $sin\left(\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

Answer 20:

Given that $sin\left(\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right)$

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore sin\left(\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right)$

$= sin\left[\frac{\pi}{3} - sin^{-1}\left(-sin\frac{\pi}{6}\right)\right]$

$= sin\left[\frac{\pi}{3} - sin^{-1}\left\{sin\left(-\frac{\pi}{6}\right)\right\}\right]$

$= sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = sin\left(\frac{3\pi}{6}\right)$

$= sin\frac{\pi}{2} = 1$

Hence, $sin\left(\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right) = 1$

Hence, the option (D) is correct.

Question 21:

$tan^{-1}\sqrt{3} - cot^{-1}(-\sqrt{3})$ is equal to

(A) π (B) $-\frac{\pi}{2}$ (C) 0 (D) $2\sqrt{3}$

Answer 21:

Given that $tan^{-1}\sqrt{3} - cot^{-1}(-\sqrt{3})$

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\cot^{-1}$ is $(0, \pi)$.

$\therefore tan^{-1}\sqrt{3} - cot^{-1}(-\sqrt{3})$

$= tan^{-1}\left(tan\frac{\pi}{3}\right) - cot^{-1}\left(-cot\frac{\pi}{6}\right)$

$= \frac{\pi}{3} - cot^{-1}\left[cot\left(\pi - \frac{\pi}{6}\right)\right]$

$= \frac{\pi}{3} - cot^{-1}\left(cot\frac{5\pi}{6}\right)$

$= \frac{\pi}{3} - \frac{5\pi}{6} = \frac{2\pi - 5\pi}{6} = -\frac{3\pi}{6} = -\frac{\pi}{2}$

Hence, $tan^{-1}\sqrt{3} - cot^{-1}(-\sqrt{3}) = -\frac{\pi}{2}$

Hence, the options (B) is correct.