Miscellaneous Exercise on Chapter 2

Question 1:

Find the value of $cos^{-1}\left(cos\frac{13\pi}{6}\right)$.

Answer 1:

Given that $cos^{-1}\left(cos\frac{13\pi}{6}\right)$

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

$\therefore cos^{-1}\left(cos\frac{13\pi}{6}\right) = cos^{-1}\left[cos\left(2\pi + \frac{\pi}{6}\right)\right]$

$= cos^{-1}\left(cos\frac{\pi}{6}\right) = \frac{\pi}{6} \in [0, \pi]$

Hence, $cos^{-1}\left(cos\frac{13\pi}{6}\right) = \frac{\pi}{6}$

Question 2:

Find the value of $tan^{-1}\left(tan\frac{7\pi}{6}\right)$.

Answer 2:

Given that $tan^{-1}\left(tan\frac{7\pi}{6}\right)$

We know that $\tan^{-1}(\tan x) = x$ if, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

$\therefore tan^{-1}\left(tan\frac{7\pi}{6}\right) = tan^{-1}\left[tan\left(\pi + \frac{\pi}{6}\right)\right]$

$= tan^{-1}\left(tan\frac{\pi}{6}\right) = \frac{\pi}{6}$

Hence, $tan^{-1}\left(tan\frac{7\pi}{6}\right) = \frac{\pi}{6}$

Question 3:

Prove that $2sin^{-1}\frac{3}{5} = tan^{-1}\frac{24}{7}$.

Answer 3:

LHS$= 2sin^{-1}\frac{3}{5} = 2tan^{-1}\frac{3}{\sqrt{5^2-3^2}}$ $\quad \left[as\ sin^{-1}\frac{a}{b} = tan^{-1}\frac{a}{\sqrt{b^2-a^2}}\right]$

$= 2tan^{-1}\frac{3}{4} = tan^{-1}\left[\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right]$ $\quad \left[as\ 2tan^{-1}x = tan^{-1}\frac{2x}{1-x^2}\right]$

$= tan^{-1}\left[\frac{\frac{3}{2}}{\frac{16-9}{16}}\right] = tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right) = tan^{-1}\frac{24}{7}$ = RHS

Question 4:

Prove that $sin^{-1}\frac{8}{17} + sin^{-1}\frac{3}{5} = tan^{-1}\frac{77}{36}$.

Answer 4:

LHS $= sin^{-1}\frac{8}{17} + sin^{-1}\frac{3}{5}$

$= tan^{-1}\frac{8}{\sqrt{17^2-8^2}} + tan^{-1}\frac{3}{\sqrt{5^2-3^2}}$ $\quad \left[as\ sin^{-1}\frac{a}{b} = tan^{-1}\frac{a}{\sqrt{b^2-a^2}}\right]$

$= tan^{-1}\frac{8}{15} + tan^{-1}\frac{3}{4}$

$= tan^{-1}\left[\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}\right]$ $\left[as\ tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= tan^{-1}\left[\frac{\frac{32+45}{15\times4}}{\frac{15\times4-8\times3}{15\times4}}\right] = tan^{-1}\left[\frac{\frac{77}{60}}{\frac{36}{60}}\right]$

$= tan^{-1}\frac{77}{36} = \text{RHS}$

Question 5:

Prove that $cos^{-1}\frac{4}{5} + cos^{-1}\frac{12}{13} = cos^{-1}\frac{33}{65}$

Answer 5:

$\text{LHS} = cos^{-1}\frac{4}{5} + cos^{-1}\frac{12}{13}$

$= tan^{-1}\frac{\sqrt{5^2-4^2}}{4} + tan^{-1}\frac{\sqrt{13^2-12^2}}{12}$ $\left[as\ cos^{-1}\frac{a}{b} = tan^{-1}\frac{\sqrt{b^2-a^2}}{a}\right]$

$= tan^{-1}\frac{3}{4} + tan^{-1}\frac{5}{12}$

$= tan^{-1}\left[\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}}\right]$ $\left[as\ tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= tan^{-1}\left[\frac{\frac{36+20}{4\times12}}{\frac{4\times12-3\times5}{4\times12}}\right] = tan^{-1}\frac{56}{33}$

$= cos^{-1}\frac{33}{\sqrt{56^2+33^2}}$ $\left[as\ tan^{-1}\frac{a}{b} = cos^{-1}\frac{b}{\sqrt{a^2+b^2}}\right]$

$= cos^{-1}\frac{33}{\sqrt{4225}} = cos^{-1}\frac{33}{65} = \text{RHS}$

Question 6:

Prove that $cos^{-1}\frac{12}{13} + sin^{-1}\frac{3}{5} = sin^{-1}\frac{56}{65}$

Answer 6:

$\text{LHS} = cos^{-1}\frac{12}{13} + sin^{-1}\frac{3}{5}$

$= tan^{-1}\frac{\sqrt{13^2-12^2}}{12} + tan^{-1}\frac{3}{\sqrt{5^2-3^2}}$

$\left[as\ cos^{-1}\frac{a}{b} = tan^{-1}\frac{\sqrt{b^2-a^2}}{a} \text{ and } sin^{-1}\frac{a}{b} = tan^{-1}\frac{a}{\sqrt{b^2-a^2}}\right]$

$= tan^{-1}\frac{5}{12} + tan^{-1}\frac{3}{4}$

$= tan^{-1}\left[\frac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}\times\frac{3}{4}}\right]$ $\left[as\ tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$$= tan^{-1}\left[\frac{\frac{20+36}{12\times 4}}{\frac{12\times 4-5\times 3}{12\times 4}}\right] = tan^{-1}\frac{56}{33}$$

$$= sin^{-1}\frac{56}{\sqrt{56^2+33^2}} \qquad \left[as\ tan^{-1}\frac{a}{b} = sin^{-1}\frac{a}{\sqrt{a^2+b^2}}\right]$$

$$= sin^{-1}\frac{56}{\sqrt{4225}} = sin^{-1}\frac{56}{65} = \text{RHS}$$

Question 7:

Prove that $tan^{-1}\frac{63}{16} = sin^{-1}\frac{5}{13} + cos^{-1}\frac{3}{5}$

Answer 7:

$$\text{RHS} = sin^{-1}\frac{5}{13} + cos^{-1}\frac{3}{5}$$

$$= tan^{-1}\frac{5}{\sqrt{13^2-5^2}} + tan^{-1}\frac{\sqrt{5^2-3^2}}{3} \quad \left[as\ cos^{-1}\frac{a}{b} = tan^{-1}\frac{\sqrt{b^2-a^2}}{a} \text{ and } sin^{-1}\frac{a}{b} = tan^{-1}\frac{a}{\sqrt{b^2-a^2}}\right]$$

$$= tan^{-1}\frac{5}{12} + tan^{-1}\frac{4}{3}$$

$$= tan^{-1}\left[\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right] \qquad \left[as\ tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= tan^{-1}\left[\frac{\frac{15+48}{12\times 3}}{\frac{12\times 3-5\times 4}{12\times 3}}\right] = tan^{-1}\frac{63}{16} = \text{RHS}$$

Question 8:

Prove that $tan^{-1}\frac{1}{5} + tan^{-1}\frac{1}{7} + tan^{-1}\frac{1}{3} + tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Answer 8:

$$\text{LHS} = tan^{-1}\frac{1}{5} + tan^{-1}\frac{1}{7} + tan^{-1}\frac{1}{3} + tan^{-1}\frac{1}{8}$$

$$= tan^{-1}\left[\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right] + tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right] \qquad \left[as\ tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= tan^{-1}\left[\frac{\frac{7+5}{5\times 7}}{\frac{5\times 7-1\times 1}{5\times 7}}\right] + tan^{-1}\left[\frac{\frac{8+3}{3\times 8}}{\frac{3\times 8-1\times 1}{3\times 8}}\right]$$

$$= tan^{-1}\frac{12}{34} + tan^{-1}\frac{11}{23} = tan^{-1}\frac{6}{17} + tan^{-1}\frac{11}{23}$$

$$= tan^{-1}\left[\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right] \qquad \left[as\ tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= tan^{-1}\left[\frac{\frac{138+187}{17\times 23}}{\frac{17\times 23-6\times 11}{17\times 23}}\right] = tan^{-1}\left(\frac{138+187}{391-66}\right)$$

$$= tan^{-1}\frac{325}{325} = tan^{-1}1 = \frac{\pi}{4} = \text{RHS}$$

Question 9:

Prove that $tan^{-1}\sqrt{x} = \frac{1}{2}cos^{-1}\left(\frac{1-x}{1+x}\right), x \in [0, 1]$

Answer 9:

LHS $= tan^{-1}\sqrt{x} = \frac{1}{2} \times 2tan^{-1}\sqrt{x} = \frac{1}{2} \times 2tan^{-1}\sqrt{x}$

$= \frac{1}{2}cos^{-1}\left[\frac{1-(\sqrt{x})^2}{1+(\sqrt{x})^2}\right]$ $\quad \left[as\ 2tan^{-1}x = cos^{-1}\left[\frac{1-x^2}{1+x^2}\right]\right]$

$= \frac{1}{2}cos^{-1}\left(\frac{1-x}{1+x}\right)$ = RHS

Question 10:

Prove that $cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$

Answer 10:

LHS $= cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = cot^{-1}\left(\frac{\sqrt{1+\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{1-\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{1+\cos\left(\frac{\pi}{2}-x\right)}-\sqrt{1-\cos\left(\frac{\pi}{2}-x\right)}}\right)$

$= cot^{-1}\left(\frac{\sqrt{1+\cos y}+\sqrt{1-\cos y}}{\sqrt{1+\cos y}-\sqrt{1-\cos y}}\right)$ $\quad \left[Let\ \frac{\pi}{2} - x = y\right]$

$= cot^{-1}\left(\frac{\sqrt{2cos^2\frac{y}{2}}+\sqrt{2sin^2\frac{y}{2}}}{\sqrt{2cos^2\frac{y}{2}}-\sqrt{2sin^2\frac{y}{2}}}\right)$ $\quad \left[as\ 1+\cos y = 2cos^2\frac{y}{2} and\ 1 - cosy = \ 2sin^2\frac{y}{2}\right]$

$= cot^{-1}\left(\frac{\sqrt{2}cos\frac{y}{2}+\sqrt{2}sin\frac{y}{2}}{\sqrt{2}cos\frac{y}{2}-\sqrt{2}sin\frac{y}{2}}\right)$

$= cot^{-1}\left(\frac{1+tan\frac{y}{2}}{1-tan\frac{y}{2}}\right)$ $\quad \left[Dividing\ each\ term\ by\ \sqrt{2}cos\frac{y}{2}\right]$

$= cot^{-1}\left(\frac{tan\frac{\pi}{4}+tan\frac{y}{2}}{1-tan\frac{\pi}{4}.tan\frac{y}{2}}\right) = cot^{-1}\left[tan\left(\frac{\pi}{4}+\frac{y}{2}\right)\right]$

$= cot^{-1}\left[cot\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}+\frac{y}{2}\right)\right\}\right] = \frac{\pi}{2} - \left(\frac{\pi}{4}+\frac{y}{2}\right) = \frac{\pi}{4} - \frac{y}{2}$

$= \frac{\pi}{4} - \frac{1}{2}\left(\frac{\pi}{2} - x\right)$ $\quad \left[as\ \frac{\pi}{2} - x = y\right]$

$= \frac{x}{2}$ = RHS

Question 11:

Prove that $tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}cos^{-1}x, -\frac{1}{\sqrt{2}} \le x \le 1.$

Answer 11:

LHS $= tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$= tan^{-1}\left(\frac{\sqrt{1+\cos y}-\sqrt{1-\cos y}}{\sqrt{1+\cos y}+\sqrt{1-\cos y}}\right)$ [Let $x = \cos y$]

$= tan^{-1}\left(\frac{\sqrt{2cos^2\frac{y}{2}}-\sqrt{2sin^2\frac{y}{2}}}{\sqrt{2cos^2\frac{y}{2}}+\sqrt{2sin^2\frac{y}{2}}}\right)$ $\left[as\ 1+\cos y = 2cos^2\frac{y}{2} and\ 1-cosy = 2sin^2\frac{y}{2}\right]$

$= tan^{-1}\left(\frac{\sqrt{2}cos\frac{y}{2}-\sqrt{2}sin\frac{y}{2}}{\sqrt{2}cos\frac{y}{2}+\sqrt{2}sin\frac{y}{2}}\right)$

$= tan^{-1}\left(\frac{1-tan\frac{y}{2}}{1+tan\frac{y}{2}}\right)$ $\left[Dividing\ each\ term\ by\ \sqrt{2}cos\frac{y}{2}\right]$

$= tan^{-1}\left(\frac{tan\frac{\pi}{4}-tan\frac{y}{2}}{1+tan\frac{\pi}{4}.tan\frac{y}{2}}\right) = tan^{-1}\left[tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]$

$= \frac{\pi}{4}-\frac{y}{2} = \frac{\pi}{4}-\frac{1}{2}cos^{-1}x = \text{RHS}$

Question 12:

Prove that $\frac{9\pi}{8}-\frac{9}{4}sin^{-1}\frac{1}{3} = \frac{9}{4}sin^{-1}\frac{2\sqrt{2}}{3}$

Answer 12:

$\text{LHS} = \frac{9\pi}{8}-\frac{9}{4}sin^{-1}\frac{1}{3} = \frac{9}{4}\left(\frac{\pi}{2}-sin^{-1}\frac{1}{3}\right)$

$= \frac{9}{4}\left(cos^{-1}\frac{1}{3}\right)$ $\left[as\ sin^{-1}x+cos^{-1}x = \frac{\pi}{2}\right]$

$= \frac{9}{4}\left(sin^{-1}\frac{\sqrt{3^2-1^2}}{3}\right)$ $\left[as\ cos^{-1}\frac{a}{b} = sin^{-1}\frac{\sqrt{b^2-a^2}}{b}\right]$

$= \frac{9}{4}\left(sin^{-1}\frac{\sqrt{8}}{3}\right) = \frac{9}{4}\left(sin^{-1}\frac{2\sqrt{2}}{3}\right) = \text{RHS}$

Question 13:

Solve for x: $2tan^{-1}(\cos x) = tan^{-1}(2\operatorname{cosec} x)$

Answer 13:

Given that $2tan^{-1}(\cos x) = tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow tan^{-1}\left(\frac{2\cos x}{1-cos^2x}\right) = tan^{-1}(2\operatorname{cosec} x)$ $\left[as\ 2tan^{-1}x = tan^{-1}\frac{2x}{1-x^2}\right]$

$\Rightarrow \frac{2\cos x}{1-cos^2x} = 2\operatorname{cosec} x$

$\Rightarrow \frac{2\cos x}{sin^2x} = \frac{2}{\sin x}$ $\Rightarrow 2\sin x.\cos x = 2sin^2x$

$\Rightarrow 2\sin x.\cos x - 2sin^2x = 0$ $\Rightarrow 2\sin x(\cos x-\sin x) = 0$

$\Rightarrow 2\sin x = 0$ *or* $\cos x - \sin x = 0$

But $\sin x \neq 0$ as it does not satisfy the equation

$\therefore \cos x - \sin x = 0$ $\Rightarrow \cos x = \sin x$ $\Rightarrow \tan x = 1$

$\therefore x = \frac{\pi}{4}$

Question 14:

Solve for x: $tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}tan^{-1}x,\ (x > 0)$

Answer 14:

Given that $tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}tan^{-1}x$

$\Rightarrow tan^{-1}1 - tan^{-1}x = \frac{1}{2}tan^{-1}x$ $\left[\because\ tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$

$\Rightarrow \frac{\pi}{4} = \frac{3}{2}tan^{-1}x$ $\Rightarrow \frac{\pi}{6} = tan^{-1}x$

$\Rightarrow tan\left(\frac{\pi}{6}\right) = x$

$\therefore x = \frac{1}{\sqrt{3}}$

Question 15:

$sin(tan^{-1}x), |x| < 1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$ (C) $\frac{1}{\sqrt{1+x^2}}$ (D) $\frac{x}{\sqrt{1+x^2}}$

Answer 15:

Given that: $sin(tan^{-1}x)$

$= sin\left(sin^{-1}\frac{x}{\sqrt{1+x^2}}\right)$ $\left[as\ tan^{-1}\frac{a}{b} = sin^{-1}\frac{a}{\sqrt{a^2+b^2}}\right]$

$= \frac{x}{\sqrt{1+x^2}}$

Hence, the option (D) is correct.

Question 16:

$sin^{-1}(1-x) - 2sin^{-1}x = \frac{\pi}{2}$, then x is equal to

(A) 0, $\frac{1}{2}$ (B) 1, $\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$

Answer 16:

Given that $sin^{-1}(1-x) - 2sin^{-1}x = \frac{\pi}{2}$

Let $x = sin\ y$

$\therefore sin^{-1}(1-\sin y) - 2y = \frac{\pi}{2}$

$\Rightarrow sin^{-1}(1-\sin y) = \frac{\pi}{2} + 2y$

$\Rightarrow 1 - \sin y = sin\left(\frac{\pi}{2} + 2y\right)$

$\Rightarrow 1 - \sin y = \cos 2y$

$\Rightarrow 1 - \sin y = 1 - 2sin^2y$ $[as\ \cos 2y = 1 - 2sin^2y]$

$\Rightarrow 2sin^2y - \sin y = 0$

$\Rightarrow 2x^2 - x = 0$ $[as\ x\ =\ sin\ y]$

$\Rightarrow x(2x-1) = 0$

$\Rightarrow x = 0\quad OR\quad x = \frac{1}{2}$

But $x \neq \frac{1}{2}$, as it does not satisfy the given equation.

$\therefore x = 0$ is the solution of the given equation.

Hence, the option (C) is correct.

Question 17:

$tan^{-1}\left(\frac{x}{y}\right) - tan^{-1}\frac{x-y}{x+y}$ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $-\frac{3\pi}{4}$

Answer 17:

$$tan^{-1}\left(\frac{x}{y}\right) - tan^{-1}\frac{x-y}{x+y}$$

$$= tan^{-1}\left[\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \frac{x}{y} \times \frac{x-y}{x+y}}\right] \qquad \left[as\ tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= tan^{-1}\left[\frac{\frac{x(x+y) - y(x-y)}{y(x+y)}}{\frac{y(x+y) + x(x-y)}{y(x+y)}}\right]$$

$$= tan^{-1}\left[\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy}\right]$$

$$= tan^{-1}\left[\frac{x^2 + y^2}{x^2 + y^2}\right]$$

$$= tan^{-1}1 = \frac{\pi}{4}$$

Hence, the option (C) is correct.