Exercise 4.2

Using the property of determinants and without expanding in Exercises 1 to 7, prove that:

Question 1:

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$$

Answer 1:

$$\text{LHS} = \begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = \begin{vmatrix} x+a & a & x+a \\ y+b & b & y+b \\ z+c & c & z+c \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2]$$

$= 0 = \text{RHS}$ $[\because\ C_1 = C_3]$

Question 2:

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Answer 2:

$$\text{LHS} = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$= 0 = \text{RHS}$ $[\because$ In column C_1 every element is zero.$]$

Question 3:

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$$

Answer 3:

$$\text{LHS} = \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = \begin{vmatrix} 2 & 7 & 63 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix} \quad [\text{Applying } C_3 \to C_3 - C_1]$$

$$= 9\begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix} \quad [\text{Taking common 9 from } C_3]$$

$= 0 = \text{RHS}$ $[\because\ C_2 = C_3]$

Question 4:

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Answer 4:

$$\text{LHS} = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix} \quad [\text{Applying } C_3 \to C_3 + C_2]$$

$= (ab+bc+ca)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$ [Taking $ab+bc+ca$ as common from C_3]

$= 0 = \text{RHS}$ [$\because\ C_1 = C_3$]

Question 5:

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Answer 5:

$\text{LHS} = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$

$= \begin{vmatrix} 2c & 2r & 2z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$ [Applying $R_1 \to R_1 + R_2 - R_3$]

$= 2\begin{vmatrix} c & r & z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$ [Taking 2 as common from R_1]

$= 2\begin{vmatrix} c & r & z \\ a & p & x \\ a+b & p+q & x+y \end{vmatrix}$ [Applying $R_2 \to R_2 - R_1$]

$= 2\begin{vmatrix} c & r & z \\ a & p & x \\ b & q & y \end{vmatrix}$ [Applying $R_3 \to R_3 - R_2$]

$= -2\begin{vmatrix} a & p & x \\ c & r & z \\ b & q & y \end{vmatrix}$ [Applying $R_1 \leftrightarrow R_2$]

$= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \text{RHS}$ [Applying $R_2 \leftrightarrow R_3$]

By using properties of determinants, in Exercises 8 to 14, show that:

Question 6:

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Answer 6:

$\text{LHS} = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = \begin{vmatrix} 0 & a & -b \\ -ab & 0 & -bc \\ ab & ac & 0 \end{vmatrix}$ [Applying $R_2 \to bR_2$ and $R_3 \to aR_3$]

$= \begin{vmatrix} 0 & a & -b \\ 0 & ac & -bc \\ ab & ac & 0 \end{vmatrix}$ [Applying $R_2 \to R_2 + R_3$]

$= ab(-abc + abc)$ [Expanding along C_1]

$= ab(0) = 0 = RHS$

Question 7:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Answer 7:

$$\text{LHS} = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$ [Taking a, b, c as common from C_1, C_2, C_3]

$$= a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$ [Taking a, b, c as common from R_1, R_2, R_3]

$$= a^2b^2c^2\begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$ [Applying $C_1 \to C_1 + C_2$]

$= a^2b^2c^2\{2(1+1)\}$ [Expanding along C_1]

$= 4a^2b^2c^2 = \text{RHS}$

Question 8:

(i) $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

(ii) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Answer 8:

(i) LHS $$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$ [Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$]

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$ [Taking common $\mathrm{a-b}$ from $\mathrm{R_1}$ and $\mathrm{b-c}$ from $\mathrm{R_2}$]

$= (a-b)(b-c)\{1(b+c-a-b)\}$ [Expanding along C_1]

$= (a-b)(b-c)(c-a) = \text{RHS}$

(ii) LHS $$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$ [By $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$]

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

[Taking common $\mathrm{a-b}$ from $\mathrm{C_1}$ and $\mathrm{b-c}$ from $\mathrm{C_2}$]

$= (a-b)(b-c)\{1(b^2+bc+c^2) - (a^2+ab+b^2)\}$ [Expanding along R_1]

$= (a-b)(b-c)\{c^2-a^2+bc-ab\}$

$= (a-b)(b-c)\{(c-a)(c+a)+b(c-a)\}$

$= (a-b)(b-c)(c-a)\{c+a+b\}$

$= (a-b)(b-c)(c-a)(a+b+c) = \text{RHS}$

Question 9:

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Answer 9:

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = \begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix} \quad [\text{Applying } R_1 \to xR_1, R_2 \to yR_2, R_3 \to zR_3]$$

$$= xyz \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix} \quad [\text{Taking } xyz \text{ as common from } C_3]$$

$$= xyz \begin{vmatrix} x^2-y^2 & x^3-y^3 & 0 \\ y^2-z^2 & y^3-z^3 & 0 \\ z^2 & z^3 & 1 \end{vmatrix} \quad [\text{Applying } R_1 \to R_1 - R_2, R_2 \to R_2 - R_3]$$

$$= xyz(x-y)(y-z) \begin{vmatrix} x+y & x^2+xy+y^2 & 0 \\ y+z & y^2+yz+z^2 & 0 \\ z^2 & z^3 & 1 \end{vmatrix}$$

[Taking $x - y$ as common from R_1 and $y - z$ from R_2]

$$= xyz(x-y)(y-z)\{(x+y)(y^2+yz+z^2) - (y+z)(x^2+xy+y^2)\}$$

[Expanding along C_3]

$$= xyz(x-y)(y-z)\{xy^2+xyz+xz^2+y^3+y^2z+yz^2-(x^2y+xy^2+y^3+x^2z+xyz+y^2z)\}$$

$$= xyz(x-y)(y-z)\{xz^2+yz^2-x^2y-x^2z\} = xyz(x-y)(y-z)\{xz^2-x^2z+yz^2-x^2y\}$$

$$= xyz(x-y)(y-z)\{xz(z-x)+y(z^2-x^2)\}$$

$$= xyz(x-y)(y-z)(z-x)\{xz+y(z+x)\}$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx) = \text{RHS}$$

Question 10:

(i) $$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

(ii) $$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$$

Answer 10:

(i) LHS $$= \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= \begin{vmatrix} 5x+4 & 2x & 2x \\ 5x+4 & x+4 & 2x \\ 5x+4 & 2x & x+4 \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= (5x+4) \begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix} \quad [\text{Taking } 5x+4 \text{ as common from } C_1]$$

$$= (5x+4) \begin{vmatrix} 0 & x-4 & 0 \\ 0 & 4-x & x-4 \\ 1 & 2x & x+4 \end{vmatrix} \quad [\text{Applying } R_1 \to R_1 - R_2, R_2 \to R_2 - R_3]$$

$$= (5x+4)\{(x-4)(x-4) - (4-x)0\} \quad [\text{Expanding along } C_1]$$

$$= (5x+4)(4-x)^2 = \text{RHS}$$

(ii) LHS $= \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$

$= \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$= (3y+k) \begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix}$ [Taking $3y+k$ as common from C_1]

$= (3y+k) \begin{vmatrix} 0 & -k & 0 \\ 0 & k & -k \\ 1 & y & y+k \end{vmatrix}$ [Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$]

$= (3y+k)\{(-k)(-k) - (k)0\}$ [Expanding along C_1]

$= (3y+k)k^2 =$ RHS

Question 11:

(i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

Answer 11:

(i) LHS $= \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ [Applying $R_1 \to R_1 + R_2 + R_3$]

$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ [Taking $a+b+c$ as common from R_1]

$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -a-b-c & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$ [By $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$]

$= (a+b+c)\{(a+b+c)^2 - 0\}$ [Expanding along R_1]

$= (a+b+c)^3 =$ RHS

(ii) LHS $= \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$

$= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$ [Taking $2(x+y+z)$ common from C_1]

$$= 2(x+y+z)\begin{vmatrix} 0 & -(x+y+z) & 0 \\ 0 & x+y+z & -(x+y+z) \\ 1 & x & z+x+2y \end{vmatrix} \text{ [By } R_1 \to R_1 - R_2, R_2 \to R_2 - R_3]$$

$= 2(x+y+z)\{(x+y+z)^2 - 0\}$ [Expanding along C_1]

$= 2(x+y+z)^3 = \text{RHS}$

Question 12:

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

Answer 12:

$$\text{LHS} = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix} \text{ [Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix} \text{ [Taking } 1+x+x^2 \text{ as common from } C_1]$$

$$= (1+x+x^2)\begin{vmatrix} 0 & x-1 & x^2-x \\ 0 & 1-x^2 & x-1 \\ 1 & x^2 & 1 \end{vmatrix} \text{ [Applying } R_1 \to R_1 - R_2, R_2 \to R_2 - R_3]$$

$$= (1+x+x^2)(1-x)^2\begin{vmatrix} 0 & -1 & -x \\ 0 & 1+x & -1 \\ 1 & x^2 & 1 \end{vmatrix} \text{ [Taking } 1-x \text{ as common from } R_1 \text{ and } R_2]$$

$= (1+x+x^2)(1-x)^2\{1 + x(1+x)\}$ [Expanding along C_1]

$= (1+x+x^2)(1-x)^2(1+x+x^2) = (1-x^3)^2 = \text{RHS}$

Question 13:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

Answer 13:

$$\text{LHS} = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} 1+a^2-b^2 & 2ab & -2ab \\ 2ab & 1-a^2+b^2 & 2a^2 \\ 2b & -2a & a-a^3-ab^2 \end{vmatrix} \text{ [Applying } C_3 \to aC_3]$$

$$= \frac{1}{a}\begin{vmatrix} 1+a^2-b^2 & 2ab & 0 \\ 2ab & 1-a^2+b^2 & 1+a^2+b^2 \\ 2b & -2a & -a-a^3-ab^2 \end{vmatrix} \text{ [Applying } C_3 \to C_3 + C_2]$$

$$= \frac{1+a^2+b^2}{a}\begin{vmatrix} 1+a^2-b^2 & 2ab & 0 \\ 2ab & 1-a^2+b^2 & 1 \\ 2b & -2a & -a \end{vmatrix}$$

[Taking $1+a^2+b^2$ as common from C_3]

$$= \frac{1+a^2+b^2}{a^2}\begin{vmatrix} 1+a^2-b^2 & 2ab & 0 \\ 2a^2b & a-a^3+ab^2 & a \\ 2b & -2a & -a \end{vmatrix} \text{ [Applying } R_2 \to aR_2]$$

$$= \frac{1+a^2+b^2}{a^2}\begin{vmatrix} 1+a^2-b^2 & 2ab & 0 \\ 2a^2b+2b & -a-a^3+ab^2 & 0 \\ 2b & -2a & -a \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 + R_3]$$

$$= (1+a^2+b^2)\begin{vmatrix} 1+a^2-b^2 & 2b & 0 \\ 2a^2b+2b & -1-a^2+b^2 & 0 \\ 2b & -2 & -1 \end{vmatrix}$$

[Taking a as common from C_2 and C_3]

$$= (1+a^2+b^2)(-1)\{(1+a^2-b^2)(-1-a^2+b^2) - 2b(2a^2b+2b)\}$$

[Expanding along C_3]

$$= -(1+a^2+b^2)\{-1-a^2+b^2-a^2-a^4+a^2b^2+b^2+a^2b^2-b^4-4a^2b^2-4b^2\}$$

$$= (1+a^2+b^2)\{1+a^4+4+2a^2+2a^2b^2+2b^2\}$$

$$= (1+a^2+b^2)(1+a^2+b^2)^2 = (1-x^3)^2 = \text{RHS}$$

Question 14:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Answer 14:

$$\text{LHS} = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a^3+a & a^2b & a^2c \\ ab^2 & b^3+b & b^2c \\ c^2a & c^2b & c^3+c \end{vmatrix} \quad [\text{Applying } R_1 \to aR_1, R_3 \to bR_3, R_3 \to cR_3]$$

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

[Taking a as common from C_1, b from C_2 and c from C_3]

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

[By $R_1 \to R_1 + R_2 + R_3$]

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

[Taking $1+a^2+b^2+c^2$ as common from R_1]

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & b^2 \\ 0 & -1 & c^2+1 \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 - C_2, C_2 \to C_2 - C_3]$$

$$= (1+a^2+b^2+c^2)\{1-0\} \quad [\text{Expanding along } R_1]$$

$$= 1+a^2+b^2+c^2 = \text{RHS}$$

Choose the correct answer in Exercises 15 and 16.

Question 15:

Let A be a square matrix of order 3 × 3, then $|kA|$ is equal to:

(A) $k|A|$ (B) $k^2|A|$ (C) $k^3|A|$ (D) $3k|A|$

Answer 15:

If B be a square matrix of order $n \times n$, then $|kB| = k^{n-1}|B|$

Therefore, $|kA| = k^{3-1}|A| = k^2|A|$

Hence, the option (B) is correct.

Question 16:

Which of the following is correct

(A) Determinant is a square matrix.

(B) Determinant is a number associated to a matrix.

(C) Determinant is a number associated to a square matrix.

(D) None of these

Answer 16:

Determinant is a number associated to a square matrix.

Hence, the option (C) is correct.