Exercise 5.3

Find $\frac{dy}{dx}$ in the following:

Question 1:

$2x + 3y = \sin x$

Answer 1:

$$2x + 3y = \sin x$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}\sin x$$

$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos x \qquad \Rightarrow \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

Question 2:

$2x + 3y = \sin y$

Answer 2:

$$2x + 3y = \sin y$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}\sin y \qquad \Rightarrow 2 + 3\frac{dy}{dx} = \cos y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(\cos y - 3) = 2 \qquad \Rightarrow \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

Question 3:

$ax + by^2 = \cos y$

Answer 3:

$$ax + by^2 = \cos y$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}\cos y \qquad \Rightarrow a + 2by\frac{dy}{dx} = -\sin y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(2by + \sin y) = -a \qquad \Rightarrow \frac{dy}{dx} = -\frac{a}{2by + \sin y}$$

Question 4:

$xy + y^2 = \tan x + y$

Answer 4:

$$xy + y^2 = \tan x + y$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}\tan x + \frac{dy}{dx}$$

$$\Rightarrow x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(x + 2y - 1) = \sec^2 x - y \qquad \Rightarrow \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}$$

Question 5:

$x^2 + xy + y^2 = 100$

Answer 5:

$$x^2 + xy + y^2 = 100$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}x^2 + \frac{d}{dx}(xy) + \frac{d}{dx}y^2 = \frac{d}{dx}(100)$$

$$\Rightarrow 2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(x + 2y) = 2x + y \qquad \Rightarrow \frac{dy}{dx} = \frac{2x + y}{x + 2y}$$

Question 6:

$x^3 + x^2y + xy^2 + y^3 = 81$

Answer 6:

$$x^3 + x^2y + xy^2 + y^3 = 81$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}x^3 + \frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) + \frac{d}{dx}y^3 = \frac{d}{dx}81$$

$$\Rightarrow 3x^2 + x^2\frac{dy}{dx} + y.2x + x.2y\frac{dy}{dx} + y^2.1 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(x^2 + 2xy + 3y^2) = -(3x^2 + 2xy + y^2) \qquad \Rightarrow \frac{dy}{dx} = -\frac{3x^2 + 2xy + y^2}{x^2 + 2xy + 3y^2}$$

Question 7:

$\sin^2 y + \cos xy = k$

Answer 7:

$$\sin^2 y + \cos xy = k$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}\sin^2 y + \frac{d}{dx}\cos xy = \frac{d}{dx}k$$

$$\Rightarrow 2\sin y\cos y\frac{dy}{dx} - \sin xy\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \sin 2y\frac{dy}{dx} - x\sin xy\frac{dy}{dx} - y\sin xy = 0$$

$$\Rightarrow (\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Question 8:

$\sin^2 x + \cos^2 y = 1$

Answer 8:

$$\sin^2 x + \cos^2 y = 1$$

Differentiating both sides with respect to x, we get

$\frac{d}{dx}\sin^2 x + \frac{d}{dx}\cos^2 y = \frac{d}{dx}1$

$\Rightarrow 2\sin x\cos x + 2\cos y\,(-\sin y)\frac{dy}{dx} = 0$

$\Rightarrow \sin 2x - \sin 2y\frac{dy}{dx} = 0 \qquad \Rightarrow \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$

Question 9:

$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Answer 9:

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Let, $x = \tan\theta$

Therefore, $y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$

$\Rightarrow y = 2\tan^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

Question 10:

$y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Answer 10:

$$y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

Let, $x = \tan\theta$

Therefore, $y = \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}x$

$\Rightarrow y = 3\tan^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{3}{1+x^2}$$

Question 11:

$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Answer 11:

$$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Let, $x = \tan\theta$

Therefore, $y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = 2\theta = 2\tan^{-1}x$

$\Rightarrow y = 2\tan^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

Question 12:

$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Answer 12:

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Let, $x = \tan\theta$

Therefore,

$y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}(\cos 2\theta) = \sin^{-1}\left\{\sin(\frac{\pi}{2} - 2\theta)\right\} = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\tan^{-1}x$

$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = 0 - \frac{2}{1+x^2} = -\frac{2}{1+x^2}$$

Question 13:

$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$

Answer 13:

$$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

Let, $x = \tan\theta$

Therefore, $y = \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(\sin 2\theta) = \cos^{-1}\left\{\cos(\frac{\pi}{2} - 2\theta)\right\} = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\tan^{-1}x$

$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = 0 - \frac{2}{1+x^2} = -\frac{2}{1+x^2}$$

Question 14:

$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Answer 14:

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

Let, $x = \sin\theta$

Therefore, $y = \sin^{-1}\left(2\sin\theta\sqrt{1-sin^2\theta}\right)$

$= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1}x$

$\Rightarrow y = 2\sin^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Question 15:
$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Answer 15:

$$y = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$$

Let, $x = \cos\theta$

Therefore, $y = \sec^{-1}\left(\frac{1}{2cos^2\theta-1}\right) = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right) = \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1}x$

$\Rightarrow y = 2\cos^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$$