Exercise 5.6

If x and y are connected parametrically by the equations given in Exercises 1 to 10, without eliminating the parameter, Find $\frac{dy}{dx}$.

Question 1:

$x = 2at^2 , y = at^4$

Answer 1:

Here, $x = 2at^2 , y = at^4$

Therefore, $\frac{dx}{dt} = 2a(2t)$ and $\frac{dy}{dt} = a(4t^3)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4at^3}{4at} = t^2$$

Question 2:

$x = a\cos\theta , y = b\cos\theta$

Answer 2:

Here, $x = a\cos\theta , y = b\cos\theta$

Therefore, $\frac{dx}{d\theta} = a(-\sin\theta)$ and $\frac{dy}{d\theta} = b(-\sin\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

Question 3:

$x = \sin t , y = \cos 2t$

Answer 3:

Here, $x = \sin t , y = \cos 2t$

Therefore, $\frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = -\sin 2t \,.\, 2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin 2t}{\cot t} = -\frac{2(2\sin t\cos t)}{\cos t} = -4\sin t$$

Question 4:

$x = 4t, y = \frac{4}{t}$

Answer 4:

Here, $x = 4t , y = \frac{4}{t}$

Therefore, $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = -\frac{4}{t^2}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{4}{t^2}}{4} = -\frac{1}{t^2}$$

Question 5:

$x = \cos\theta - \cos 2\theta\,,\, y = \sin\theta - \sin 2\theta$

Answer 5:

Here, $x = \cos\theta - \cos 2\theta\,,\, y = \sin\theta - \sin 2\theta$

Therefore, $\frac{dx}{d\theta} = -\sin\theta + 2\sin 2\theta$ and $\frac{dy}{d\theta} = \cos\theta - 2\cos 2\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta - 2\cos 2\theta}{-\sin\theta + 2\sin 2\theta}$$

Question 6:

$x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$

Answer 6:

Here, $x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$

Therefore, $\frac{dx}{d\theta} = a(1 - \cos\theta)$ and $\frac{dy}{d\theta} = a(0 - \sin\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta)}{a(1 - \cos\theta)} = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Question 7:

$x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

Answer 7:

Here, $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

Therefore, $\frac{dx}{dt} = \frac{\sin^3 t\frac{d}{dt}\sqrt{\cos 2t} - \sqrt{\cos 2t}\frac{d}{dt}\sin^3 t}{\left(\sqrt{\cos 2t}\right)^2}$

$$= \frac{\sin^3 t.\frac{1}{2\sqrt{\cos 2t}}.(-\sin 2t).2 - \sqrt{\cos 2t}.3\sin^2 t\cos t}{\cos 2t} = \frac{-\sin^3 t.\sin 2t - 3\cos 2t.\sin^2 t\cos t}{\cos 2t\sqrt{\cos 2t}}$$

and $\frac{dy}{dt} = \frac{\cos^3 t\frac{d}{dt}\sqrt{\cos 2t} - \sqrt{\cos 2t}\frac{d}{dt}\cos^3 t}{\left(\sqrt{\cos 2t}\right)^2}$

$$= \frac{\cos^3 t.\frac{1}{2\sqrt{\cos 2t}}.(-\sin 2t).2 - \sqrt{\cos 2t}.3\cos^2 t(-\sin t)}{\cos 2t} = \frac{-\cos^3 t.\sin 2t + 3\cos 2t.\cos^2 t\sin t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\cos^3 t\,.\sin 2t + 3\cos 2t\,.\cos^2 t\sin t}{-\sin^3 t\,.\sin 2t - 3\cos 2t\,.\sin^2 t\cos t}$$

$$= \frac{-\cos^3 t\,.(2\sin t\cos t) + 3\cos 2t\,.\cos^2 t\sin t}{-\sin^3 t\,.(2\sin t\cos t) - 3\cos 2t\,.\sin^2 t\cos t} = \frac{\cos^2 t\sin t\,(-2\cos^2 t + 3\cos 2t)}{\sin^2 t\cos t\,(-2\sin^2 t - 3\cos 2t)}$$

$$= \frac{\cos t\,[-2\cos^2 t + 3(2\cos^2 t - 1)]}{\sin t\,[-2\sin^2 t - 3(1 - 2\sin^2 t)]} = \frac{\cos t\,[-2\cos^2 t + 6\cos^2 t - 3]}{\sin t\,[-2\sin^2 t - 3 + 6\sin^2 t]}$$

$$= \frac{\cos t\,[4\cos^2 t - 3]}{\sin t\,[-3 + 4\sin^2 t]} = -\frac{4\cos^3 t - 3\cos t}{3\sin t - 4\sin^3 t} = -\frac{\cos 3t}{\sin 3t} = -\cot 3t$$

Question 8:

$x = a\left(\cos t + \log\tan\frac{t}{2}\right)\ y = a\sin t$

Answer 8:

Here, $x = a\left(\cos t + \log\tan\frac{t}{2}\right)\ y = a\sin t$

Therefore, $\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}}.\sec^2\frac{t}{2}.\frac{1}{2}\right) = a\left(-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}.\frac{1}{\cos^2\frac{t}{2}}.\frac{1}{2}\right)$

$= a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{\sin t}\right) = a\left(\frac{-\sin^2 t+1}{\sin t}\right) = a\left(\frac{\cos^2 t}{\sin t}\right)$

and $\frac{dy}{dt} = a\cos t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t$$

Question 9:

$x = a\sec\theta\,, y = b\tan\theta$

Answer 9:

Here, $x = a\sec\theta\,, y = b\tan\theta$

Therefore, $\frac{dx}{d\theta} = a\sec\theta\tan\theta$

and $\frac{dy}{d\theta} = b\sec^2\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta} = \frac{b\left(\frac{1}{\cos\theta}\right)}{a\left(\frac{\sin\theta}{\cos\theta}\right)} = \frac{b}{a}\operatorname{cosec}\theta$$

Question 10:

$x = a\,(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$

Answer 10:

Here, $x = a\,(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$

Therefore, $\frac{dx}{d\theta} = a\,[-\sin\theta + (\theta\cos\theta + \sin\theta)] = a\theta\cos\theta$

and $\frac{dy}{d\theta} = a\,[\cos\theta - (-\theta\sin\theta + \cos\theta)] = a\theta\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Question 11:

If $x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

Answer 11:

Here, $x = \sqrt{a^{\sin^{-1} t}},\ \ y = \sqrt{a^{\cos^{-1} t}}$

Therefore,

$$\frac{dx}{dt} = \frac{1}{2\sqrt{a^{\sin^{-1} t}}}.\frac{d}{dx}a^{\sin^{-1} t} = \frac{1}{2\sqrt{a^{\sin^{-1} t}}}.a^{\sin^{-1} t}.\log a\frac{1}{\sqrt{1-t^2}}$$

$$= \frac{1}{2x}.x^2.\log a\frac{1}{\sqrt{1-t^2}} = \frac{x\log a}{\sqrt{1-t^2}}$$

and

$$\frac{dy}{dt} = \frac{1}{2\sqrt{a^{\cos^{-1} t}}}.\frac{d}{dx}a^{\cos^{-1} t} = \frac{1}{2\sqrt{a^{\cos^{-1} t}}}.a^{\cos^{-1} t}.\log a\frac{-1}{\sqrt{1-t^2}}$$

$$= \frac{1}{2y}.y^2.\log a\frac{1}{\sqrt{1-t^2}} = -\frac{y\log a}{\sqrt{1-t^2}}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{y\log a}{\sqrt{1-t^2}}}{\frac{x\log a}{\sqrt{1-t^2}}} = -\frac{y}{x}$$