Exercise 5.7

Find the second order derivatives of the functions given in Exercises 1 to 10.

Question 1:

$x^2 + 3x + 2$

Answer 1:

Let $y = x^2 + 3x + 2$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 3x + 2) = 2x + 3 \qquad \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(2x + 3) = 2$

Question 2:

x^{20}

Answer 2:

Let $y = x^{20}$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(x^{20}) = 20x^{19} \qquad \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(20x^{19}) = 380x^{18}$

Question 3:

$x.\cos x$

Answer 3:

Let $y = x.\cos x$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(x.\cos x) = x.\frac{d}{dx}\cos x + \cos x.\frac{d}{dx}x = -x\sin x + \cos x$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(-x\sin x + \cos x) = -\left(x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x\right) - \sin x$

$= -x\cos x - \sin x - \sin x = -(x\cos x + 2\sin x)$

Question 4:

$\log x$

Answer 4:

Let $y = \log x$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

Question 5:

$x^3\log x$

Answer 5:

Let $y = x^3\log x$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(x^3\log x) = x^3.\frac{d}{dx}\log x + \log x.\frac{d}{dx}x^3 = x^3.\frac{1}{x} + \log x.3x^2 = x^2 + 3x^2\log x$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(x^2 + 3x^2\log x) = 2x + 3\left(x^2\frac{d}{dx}\log x + \log x\frac{d}{dx}x^2\right)$

$= 2x + 3\left(x^2.\frac{1}{x} + \log x.2x\right) = 2x + 3x + 6x\log x = 5x + 6x\log x = x(5 + 6\log x)$

Question 6:

$e^x \sin 5x$

Answer 6:

Let $y = e^x \sin 5x$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \sin 5x) = e^x . \frac{d}{dx}\sin 5x + \sin 5x . \frac{d}{dx}e^x = e^x . \cos 5x . 5 + \sin 5x . e^x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(5e^x \cos 5x + e^x \sin 5x)$$

$$= 5\left(e^x . \frac{d}{dx}\cos 5x + \cos 5x . \frac{d}{dx}e^x\right) + \left(e^x . \frac{d}{dx}\sin 5x + \sin 5x . \frac{d}{dx}e^x\right)$$

$$= 5[e^x . (-\sin 5x) . 5 + \cos 5x . e^x] + [e^x . \cos 5x . 5 + \sin 5x . e^x]$$

$$= e^x(-25 \sin 5x + 5\cos 5x + 5\cos 5x + \sin 5x) = e^x(10\cos 5x - 24 \sin 5x)$$

Question 7:

$e^{6x} \cos 3x$

Answer 7:

Let $y = e^{6x} \cos 3x$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(e^{6x} \cos 3x) = e^{6x} . \frac{d}{dx}\cos 3x + \cos 3x . \frac{d}{dx}e^{6x}$$

$$= e^{6x} . (-\sin 3x) . 3 + \cos 3x . e^{6x} . 6 = 3e^{6x}(-\sin 3x + 2\cos 3x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}[3e^{6x}(-\sin 3x + 2\cos 3x)]$$

$$= 3e^{6x} . \frac{d}{dx}(-\sin 3x + 2\cos 3x) + (-\sin 3x + 2\cos 3x) . \frac{d}{dx}3e^{6x}$$

$$= 3e^{6x} . (-3\cos 3x - 6\sin 3x) + (-\sin 3x + 2\cos 3x) . 18e^{6x}$$

$$= e^{6x}(-9 \text{ cso } 3x - 18 \sin 3x - 18 \sin 3x + 36 \cos 3x)$$

$$= e^{6x}(27 \cos 3x - 36 \sin 3x)$$

$$= 9e^{6x}(3\cos 3x - 4 \sin 3x)$$

Question 8:

$\tan^{-1} x$

Answer 8:

Let $y = \tan^{-1} x$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{(1+x^2)\frac{d}{dx}1 - 1 . \frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{0 - 2x}{(1+x^2)^2} = -\frac{2x}{(1+x^2)^2}$$

Question 9:

$\log(\log x)$

Answer 9:

Let $y = \log(\log x)$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(\log(\log x)) = \frac{1}{\log x}.\frac{1}{x} = \frac{1}{x\log x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x\log x}\right) = \frac{(x\log x)\frac{d}{dx}1 - 1.\frac{d}{dx}(x\log x)}{(x\log x)^2}$$

$$= \frac{0 - \left(x.\frac{1}{x} + \log x\right)}{(x\log x)^2} = -\frac{1+\log x}{(x\log x)^2}$$

Question 10:

$\sin(\log x)$

Answer 10:

Let $y = \sin(\log x)$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(\sin(\log x)) = \cos(\log x).\frac{1}{x} = \frac{\cos(\log x)}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right] = \frac{x\frac{d}{dx}\cos(\log x) - \cos(\log x).\frac{d}{dx}x}{(x)^2}$$

$$= \frac{x\left\{-\sin(\log x).\frac{1}{x}\right\} - \cos(\log x).1}{(x)^2} = \frac{-\sin(\log x) - \cos(\log x)}{(x)^2}$$

Question 11:

If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Answer 11:

Given that: $y = 5\cos x - 3\sin x$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos x - 3\sin x) = -5\sin x - 3\cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(-5\sin x - 3\cos x) = -5\cos x + 3\sin x = -(5\cos x - 3\sin x) = -y$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$

Question 12:

If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Answer 12:

Given that: $y = \cos^{-1} x \quad \Rightarrow \cos y = x$, therefore,

$$-\sin y\frac{dy}{dx} = 1 \qquad \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\operatorname{cosec} y$$

$$\Rightarrow \frac{d^2y}{dx^2} = -(-\operatorname{cosec} y\cot y).\frac{dy}{dx} = (\operatorname{cosec} y\cot y).(-\operatorname{cosec} y) = -\operatorname{cosec}^2 y\cot y$$

Question 13:

If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$

Answer 13:

Given that: $y = 3\cos(\log x) + 4\sin(\log x)$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(3\cos(\log x) + 4\sin(\log x)) = -3\sin(\log x).\frac{1}{x} + 4\cos(\log x).\frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{d}{dx}x = \frac{d}{dx}[-3\sin(\log x) + 4\cos(\log x)]$$

$$= -3\cos(\log x).\frac{1}{x} - 4\sin(\log x).\frac{1}{x} = -\frac{1}{x}[3\cos(\log x) + 4\sin(\log x)] = -\frac{1}{x}.y$$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}y \qquad \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \qquad \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

Question 14:

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Answer 14:

Given that: $y = Ae^{mx} + Be^{nx}$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(Ae^{mx} + Be^{nx}) = mAe^{mx} + nBe^{nx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(mAe^{mx} + nBe^{nx}) = m^2Ae^{mx} + n^2Be^{nx}$$

Putting the value of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ in $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$, we get

$$\text{LHS} = (m^2Ae^{mx} + n^2Be^{nx}) - (m+n)(mAe^{mx} + nBe^{nx}) + mny$$

$$= m^2Ae^{mx} + n^2Be^{nx} - (m^2Ae^{mx} + mnBe^{nx} + mnAe^{mx} + n^2Be^{nx}) + mny$$

$$= -(mnAe^{mx} + mnBe^{nx}) + mny$$

$$= -mn(Ae^{mx} + Be^{nx}) + mny$$

$$= -mny + mny = 0 = \text{RHS}$$

Question 15:

If $y = 500e^{7x} + 600e^{-7x}$, Show that $\frac{d^2y}{dx^2} = 49y$.

Answer 15:

Given that: $y = 500e^{7x} + 600e^{-7x}$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(500e^{7x} + 600e^{-7x}) = 500e^{7x}.7 + 600e^{-7x}.(-7) = 7(500e^{7x} - 600e^{-7x})$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}7(500e^{7x} - 600e^{-7x}) = 7[500e^{7x}.7 + 600e^{-7x}.(-7)]$$

$$= 49(500e^{7x} - 600e^{-7x}) = 49y$$

$$\Rightarrow \frac{d^2y}{dx^2} = 49y$$

Question 16:

If $e^y(x+1)=1$, show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$.

Answer 16:

Given that: $e^y(x+1)=1$, therefore,

$$e^y\frac{dy}{dx}(x+1)+(x+1)\frac{d}{dx}e^y=\frac{d}{dx}1$$

$$\Rightarrow e^y+(x+1)e^y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{x+1}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{d}{dx}\left(-\frac{1}{x+1}\right)=-\left[\frac{(x+1).\frac{d}{dx}1-1.\frac{d}{dx}(x+1)}{(x+1)^2}\right]=-\left[\frac{0-1}{(x+1)^2}\right]=\frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(-\frac{1}{x+1}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

Question 17:

If $y=(\tan^{-1}x)^2$, show that $(x^2+1)^2y_2+2x(x^2+1)y_1=2$.

Answer 17:

Given that: $y=(\tan^{-1}x)^2$, therefore,

$$\frac{dy}{dx}=\frac{d}{dx}[(\tan^{-1}x)^2]=2\tan^{-1}x\,.\frac{1}{1+x^2}=\frac{2\tan^{-1}x}{1+x^2}$$

$$\Rightarrow (1+x^2)\frac{dy}{dx}=2\tan^{-1}x$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+\frac{dy}{dx}.\frac{d}{dx}(1+x^2)=\frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+\frac{dy}{dx}.2x=\frac{2}{1+x^2}$$

$$\Rightarrow (1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}=2$$

$$\Rightarrow (x^2+1)^2y_2+2x(x^2+1)y_1=2$$