Exercise 5.8

Question 1:

Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8, x \in [-4, 2]$.

Answer 1:

Given function $f(x) = x^2 + 2x - 8, x \in [-4, 2]$

(i) Function f is a polynomial function, so it is continuous in close interval $[-4, 2]$.

(ii) $f'(x) = 2x + 2$

Hence, the function f is differentiable in open interval $(-4, 2)$.

(iii) $f(-4) = (-4)^2 + 2(-4) - 8 = 16 - 8 - 8 = 0$

and $f(2) = (2)^2 + 2(2) - 8 = 4 + 4 - 8 = 0$

$\Rightarrow f(-4) = f(2)$

Here, all the three conditions of Rolle's Theorem is satisfied. Therefore, there must be a number $c \in (-4, 2)$ such that $f'(c) = 0$.

$\Rightarrow f'(c) = 2c + 2 = 0$

$\Rightarrow c = -1 \in (-4, 2)$

Hence, the Rolle's Theorem is verified for the function $f(x) = x^2 + 2x - 8, x \in [-4, 2]$.

Question 2:

Examine if Rolle's Theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's Theorem from these example?

(i) $f(x) = [x]$ for $x \in [5, 9]$

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

Answer 2:

Rolle's Theorem is applicable to function $f: [a, b] \to$ R the following three conditions of Rolle's Theorem is satisfied.

(i) Function f is continuous in closed interval $[a, b]$.

(ii) Function f is differentiable in open interval (a, b).

(iii) $f(a) = f(b)$

(i) $f(x) = [x]$ for $x \in [5, 9]$

The greatest integer function f is neither continuous in close interval $[5, 9]$ nor differentiable in open interval $(5, 9)$. Also $f(5) \neq f(9)$.

Hence, the Rolle's Theorem is not applicable to $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

The greatest integer function f is neither continuous in close interval $[-2, 2]$ nor differentiable in open interval $(2, 2)$. Also $f(-2) \neq f(2)$.

Hence, the Rolle's Theorem is not applicable to $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

The function f is a polynomial function, so it is continuous in closed interval $[1, 2]$.

$f'(x) = 2x$, hence, the function f is differentiable in open interval $(1, 2)$.

$f(1) = (1)^2 - 1 = 0$ and

$f(2) = (2)^2 - 1 = 3$,

$\Rightarrow f(1) \neq f(2)$

Hence, Rolle's Theorem is not applicable to the function $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Question 3:

If $f : [-5, 5] \to \boldsymbol{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$

Answer 3:

$f : [-5, 5] \to \boldsymbol{R}$ is a differentiable function, hence

(i) The function f is continuous in closed interval $[-5, 5]$.

(ii) The function f is continuous in open interval $(-5, 5)$.

According to Mean Value Theorem, there exists a value $c \in (-5, 5)$, such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

But it is given that $f'(x)$ does not vanish anywhere, hence

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)} \neq 0$$

$\Rightarrow f(5) - f(-5) \neq 0$

$\Rightarrow f(5) \neq f(-5)$

Question 4:

Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.

Answer 4:

Given function: $f(x) = x^2 - 4x - 3\,, x \in [1, 4]$

(i) Function f is a polynomial function, hence it is continuous in closed interval $[1, 4]$.

(ii) $f'(x) = 2x - 4$

Hence, the function f is differentiable in open interval $(1, 4)$.

According to Mean Value Theorem, there exists a value $c \in (1, 4)$, such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

$$\Rightarrow 2c - 4 = \frac{[(4)^2 - 4(4) - 3\,] - [(1)^2 - 4(1) - 3\,]}{3}$$

$$\Rightarrow 2c - 4 = \frac{-3 - (-6)}{3} = \frac{3}{3} = 1$$

$$\Rightarrow 2c = 5 \quad \Rightarrow c = \frac{5}{2} \in (1, 4)$$

Hence, for the function $f(x) = x^2 - 4x - 3\,, x \in [1, 4]$, the Mean Value Theorem is verified.

Question 5:

Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.

Answer 5:

Given function: $f(x) = x^3 - 5x^2 - 3x\,, x \in [1, 3]$

(i) Function f is a polynomial function, hence it is continuous in closed interval $[1, 3]$.

(ii) $f'(x) = 3x^2 - 10x - 3$

Hence, the function f is differentiable in open interval $(1, 3)$.

According to Mean Value Theorem, there exists a value $c \in (1, 3)$, such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$\Rightarrow 3c^2 - 10c - 3 = \dfrac{[(3)^3 - 5(3)^2 - 3(3)] - [(1)^3 - 5(1)^2 - 3(1)]}{2}$

$\Rightarrow 3c^2 - 10c - 3 = \dfrac{(27 - 54) - (1 - 8)}{2} = \dfrac{-27 + 7}{2} = -10$

$\Rightarrow 3c^2 - 10c + 7 = 0$

$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$

$\Rightarrow 3c(c - 1) - 7(c - 1) = 0$

$\Rightarrow (c - 1)(3c - 7) = 0$

$\Rightarrow c - 1 = 0 \quad \text{or} \quad 3c - 7 = 0$

$\Rightarrow c = 1 \quad \text{or} \quad c = \dfrac{7}{3}$

$\Rightarrow c = \dfrac{7}{3} \in (1, 3)$

Hence, for the function $f(x) = x^3 - 5x^2 - 3x$, $x \in [1, 3]$, the Mean Value Theorem is verified. For the value of $c = \frac{7}{3}$ the function $f'(c) = 0$.

Question 6:

Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Answer 6:

Mean Value Theorem is applicable to function $f: [a, b] \rightarrow \text{R}$ the following two conditions of Mean Value Theorem is satisfied.

(i) Function f is continuous in closed interval $[a, b]$.

(ii) Function f is differentiable in open interval (a, b).

(i) $f(x) = [x]$ for $x \in [5, 9]$

The greatest integer function f is neither continuous in close interval $[5, 9]$ nor differentiable in open interval $(5, 9)$.

Hence, the Mean Value Theorem is not applicable to $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

The greatest integer function f is neither continuous in close interval $[-2, 2]$ nor differentiable in open interval $(2, 2)$.

Hence, the Mean Value Theorem is not applicable to $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

The function f is a polynomial function, so it is continuous in closed interval $[1, 2]$.

$f'(x) = 2x$, hence, the function f is differentiable in open interval $(1, 2)$.

Hence, Mean Value Theorem is not applicable to the function $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Hence, the Mean Value Theorem is applicable to $f(x) = x^2 - 1$ for $x \in [1, 2]$.