Exercise 8.4

Question 1:

Express the trigonometric ratios sin A, sec A and tan A in terms of cot A.

Answer 1:

We know that, $\operatorname{cosec}^2 A = 1 + \cot^2 A$

$$\frac{1}{\operatorname{cosec}^2 A} = \frac{1}{1+\cot^2 A}$$

$$\sin^2 A = \frac{1}{1+\cot^2 A}$$

$$\sin A = \pm \frac{1}{\sqrt{1+\cot^2 A}}$$

Therefore, $\sin A = \frac{1}{\sqrt{1+\cot^2 A}}$

We know that, $\tan A = \frac{\sin A}{\cos A}$

However, $\cot A = \frac{\cos A}{\sin A}$

Therefore, $\tan A = \frac{1}{\cot A}$

Also, $\sec^2 A = 1 + \tan^2 A$

$$= 1 + \frac{1}{\cot^2 A}$$

$$= \frac{\cot^2 A + 1}{\cot^2 A}$$

$$\sec A = \frac{\sqrt{\cot^2 A + 1}}{\cot A}$$

Question 2:

Write all the other trigonometric ratios of ∠A in terms of sec A.

Answer 2:

We know that,

$$\cos A = \frac{1}{\sec A}$$

Also, $\sin^2 A + \cos^2 A = 1$

$\sin^2 A = 1 - \cos^2 A$

$$\sin A = \sqrt{1 - \left(\frac{1}{\sec A}\right)^2}$$

$$= \sqrt{\frac{\sec^2 A - 1}{\sec^2 A}} = \frac{\sqrt{\sec^2 A - 1}}{\sec A}$$

$\tan^2 A + 1 = \sec^2 A$

$\tan^2 A = \sec^2 A - 1$

$$\tan A = \sqrt{\sec^2 A - 1}$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{\frac{1}{\sec A}}{\frac{\sqrt{\sec^2 A - 1}}{\sec A}}$$

$$= \frac{1}{\sqrt{\sec^2 A - 1}}$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{\sec A}{\sqrt{\sec^2 A - 1}}$$

Question 3:

Evaluate

(i) $\dfrac{\sin^2 63° + \sin^2 27°}{\cos^2 17° + \cos^2 73°}$

(ii) $\sin 25° \cos 65° + \cos 25° \sin 65°$

Answer 3:

(i) $$\frac{\sin^2 63° + \sin^2 27°}{\cos^2 17° + \cos^2 73°} = \frac{\left[\sin(90° - 27°)\right]^2 + \sin^2 27°}{\left[\cos(90° - 73°)\right]^2 + \cos^2 73°}$$

$$= \frac{\left[\cos 27°\right]^2 + \sin^2 27°}{\left[\sin 73°\right]^2 + \cos^2 73°}$$

$$= \frac{\cos^2 27° + \sin^2 27°}{\sin^2 73° + \cos^2 73°}$$

$$= \frac{1}{1}$$

= 1

(ii) sin25° cos65° + cos25° sin65°

$$= (\sin 25°)\{\cos(90° - 25°)\} + \cos 25°\{\sin(90° - 25°)\}$$

$$= (\sin 25°)(\sin 25°) + (\cos 25°)(\cos 25°)$$

= $\sin^2 25° + \cos^2 25°$

= 1 (As $\sin^2 A + \cos^2 A = 1$)

Question 4:

Choose the correct option. Justify your choice.

(i) 9 $\sec^2$ A − 9 $\tan^2$ A =

(A) 1 (B) 9 (C) 8 (D) 0

(ii) (1 + tan θ + sec θ) (1 + cot θ − cosec θ)

(A) 0 (B)1 (C) 2 (D) −1

(iii) (secA + tanA) (1 − sinA) =

(A) secA (B) sinA (C) cosecA (D) cosA

(iv) $\frac{1+\tan^2 A}{1+\cot^2 A}$

(A) $\sec^2$ A (B) −1 (C) $\cot^2$ A (D) $\tan^2$ A

Answer 4:

(i) 9 $\sec^2 A$ − 9 $\tan^2 A$

= 9 ($\sec^2 A$ − $\tan^2 A$)

= 9 (1) [As $\sec^2$ A − $\tan^2$ A = 1]

= 9

Hence, alternative (B) is correct.

(ii) (1 + tan θ + sec θ) (1 + cot θ − cosec θ)

$$=\left(1+\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}\right)\left(1+\frac{\cos\theta}{\sin\theta}-\frac{1}{\sin\theta}\right)$$

$$=\left(\frac{\cos\theta+\sin\theta+1}{\cos\theta}\right)\left(\frac{\sin\theta+\cos\ \theta-1}{\sin\theta}\right)$$

$$=\frac{(\sin\theta+\cos\ \theta)^2-(1)^2}{\sin\theta\cos\theta}$$

$$=\frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\ \theta-1}{\sin\theta\cos\ \theta}$$

$$=\frac{1+2\sin\theta\cos\ \theta-1}{\sin\theta\cos\ \theta}$$

$$=\frac{2\sin\theta\cos\ \theta}{\sin\ \theta\cos\ \theta}=2$$

Hence, alternative (C) is correct.

(iii) (secA + tanA) (1 − sinA)

$$=\left(\frac{1}{\cos A}+\frac{\sin A}{\cos A}\right)(1-\sin A)$$

$$=\left(\frac{1+\sin A}{\cos A}\right)(1-\sin A)$$

$$=\frac{1-\sin^2 A}{\cos A}=\frac{\cos^2 A}{\cos A}$$

= cosA

Hence, alternative (D) is correct.

(iv) $$\frac{1+\tan^2 A}{1+\cot^2 A}=\frac{1+\frac{\sin^2 A}{\cos^2 A}}{1+\frac{\cos^2 A}{\sin^2 A}}=\frac{\frac{\cos^2 A+\sin^2 A}{\cos^2 A}}{\frac{\sin^2 A+\cos^2 A}{\sin^2 A}}=\frac{\frac{1}{\cos^2 A}}{\frac{1}{\sin^2 A}}$$

$$=\frac{\sin^2 A}{\cos^2 A}=\tan^2 A$$

Hence, alternative (D) is correct.

Question 5:

Prove the following identities, where the angles involved are acute angles for which the expressions are defined.

Answer 5:

(i) $(\text{cosec}\theta - \cot\theta)^2 = \frac{1-\cos\theta}{1+\cos\theta}$

$$\text{L.H.S.} = (\text{cosec}\,\theta - \cot\theta)^2$$
$$= \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2$$
$$= \frac{(1-\cos\theta)^2}{(\sin\theta)^2} = \frac{(1-\cos\theta)^2}{\sin^2\theta}$$
$$= \frac{(1-\cos\theta)^2}{1-\cos^2\theta} = \frac{(1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} = \frac{1-\cos\theta}{1+\cos\theta}$$
$$= \text{R.H.S.}$$

(ii) $\frac{\cos A}{1+\sin A} + \frac{1+\sin A}{\cos A} = 2\sec A$

$$\text{L.H.S.} = \frac{\cos A}{1+\sin A} + \frac{1+\sin A}{\cos A}$$
$$= \frac{\cos^2 A + (1+\sin A)^2}{(1+\sin A)(\cos A)}$$
$$= \frac{\cos^2 A + 1 + \sin^2 A + 2\sin A}{(1+\sin A)(\cos A)}$$
$$= \frac{\sin^2 A + \cos^2 A + 1 + 2\sin A}{(1+\sin A)(\cos A)}$$
$$= \frac{1+1+2\sin A}{(1+\sin A)(\cos A)} = \frac{2+2\sin A}{(1+\sin A)(\cos A)}$$
$$= \frac{2(1+\sin A)}{(1+\sin A)(\cos A)} = \frac{2}{\cos A} = 2\sec A$$
$$= \text{R.H.S.}$$

(iii) $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \sec\theta\,\text{cosec}\theta$

$$\text{L.H.S.} = \frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta}$$

$$= \frac{\frac{\sin\theta}{\cos\theta}}{1-\frac{\cos\theta}{\sin\theta}} + \frac{\frac{\cos\theta}{\sin\theta}}{1-\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\frac{\sin\theta}{\cos\theta}}{\frac{\sin\theta-\cos\theta}{\sin\theta}} + \frac{\frac{\cos\theta}{\sin\theta}}{\frac{\cos\theta-\sin\theta}{\cos\theta}}$$

$$= \frac{\sin^2\theta}{\cos\theta(\sin\theta-\cos\theta)} + \frac{\cos^2\theta}{\sin\theta(\sin\theta-\cos\theta)}$$

$$= \frac{1}{(\sin\theta-\cos\theta)}\left[\frac{\sin^2\theta}{\cos\theta} - \frac{\cos^2\theta}{\sin\theta}\right]$$

$$= \left(\frac{1}{\sin\theta-\cos\theta}\right)\left[\frac{\sin^3\theta-\cos^3\theta}{\sin\theta\cos\theta}\right]$$

$$= \left(\frac{1}{\sin\theta-\cos\theta}\right)\left[\frac{(\sin\theta-\cos\theta)(\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta)}{\sin\theta\cos\theta}\right]$$

$$= \frac{(1+\sin\theta\ \cos\theta)}{(\sin\theta\ \cos\theta)}$$

= secθ cosec θ + 1 = R.H.S.

(iv) $\frac{1+\sec A}{\sec A} = \frac{\sin^2 A}{1-\cos A}$

$$\text{L.H.S.} = \frac{1+\sec A}{\sec A} = \frac{1+\frac{1}{\cos A}}{\frac{1}{\cos A}}$$

$$= \frac{\frac{\cos A+1}{\cos A}}{\frac{1}{\cos A}} = (\cos A+1)$$

$$= \frac{(1-\cos A)(1+\cos A)}{(1-\cos A)}$$

$$= \frac{1-\cos^2 A}{1-\cos A} = \frac{\sin^2 A}{1-\cos A}$$ = R.H.S

(v) $\dfrac{\cos A-\sin A+1}{\cos A+\sin A-1}=\text{cosec}A+\cot A$

Using the identity $\text{cosec}^2 A = 1 + \cot^2 A$

L.H.S = $\dfrac{\cos A-\sin A+1}{\cos A+\sin A-1}$

$$=\frac{\dfrac{\cos A}{\sin A}-\dfrac{\sin A}{\sin A}+\dfrac{1}{\sin A}}{\dfrac{\cos A}{\sin A}+\dfrac{\sin A}{\sin A}+\dfrac{1}{\sin A}}$$

$$=\frac{\cot A-1+\text{cosec}\,A}{\cot A+1-\text{cosec}\,A}$$

$$=\frac{\{(\cot A)-(1-\text{cosec}\,A)\}\{(\cot A)-(1-\text{cosec}\,A)\}}{\{(\cot A)+(1-\text{cosec}\,A)\}\{(\cot A)-(1-\text{cosec}\,A)\}}$$

$$=\frac{(\cot A-1+\text{cosec}\,A)^2}{(\cot A)^2-(1-\text{cosec}\,A)^2}$$

$$=\frac{\cot^2 A+1+\text{cosec}^2 A-2\cot A-2\,\text{cosec}A+2\cot A\,\text{cosec}\,A}{\cot^2 A-(1+\text{cosec}^2 A-2\text{cosec}\,A)}$$

$$=\frac{2\text{cosec}^2 A+2\cot A\,\text{cosec}\,A-2\cot A-2\text{cosec}\,A}{\cot^2 A-1-\text{cosec}^2 A+2\text{cosec}\,A}$$

$$=\frac{2\text{cosec}\,A(\text{cosec}\,A+\cot A)-2(\cot A+\text{cosec}\,A)}{\cot^2 A-\text{cosec}^2 A-1+2\text{cosec}\,A}$$

$$=\frac{(\text{cosec}\,A+\cot A)(2\text{cosec}\,A-2)}{-1-1+2\text{cosec}\,A}$$

$$=\frac{(\text{cosec}\,A+\cot A)(2\text{cosec}\,A-2)}{(2\text{cosec}\,A-2)}$$

= cosec A + cot A

= R.H.S

(vi) $\sqrt{\dfrac{1+\sin A}{1-\sin A}}=\sec A+\tan A$

$$\text{L.H.S.} = \sqrt{\frac{1+\sin A}{1-\sin A}}$$

$$= \sqrt{\frac{(1+\sin A)(1+\sin A)}{(1-\sin A)(1+\sin A)}}$$

$$= \frac{(1+\sin A)}{\sqrt{1-\sin^2 A}} \qquad = \frac{1+\sin A}{\sqrt{\cos^2 A}}$$

$$= \frac{1+\sin A}{\cos A} \qquad = \sec A + \tan A$$

$$= \text{R.H.S.}$$

(vii) $\dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$

$$\text{L.H.S.} = \frac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta}$$

$$= \frac{\sin\theta\left(1-2\sin^2\theta\right)}{\cos\theta\left(2\cos^2\theta - 1\right)}$$

$$= \frac{\sin\theta \times \left(1-2\sin^2\theta\right)}{\cos\theta \times \left\{2\left(1-\sin^2\theta\right)-1\right\}}$$

$$= \frac{\sin\theta \times \left(1-2\sin^2\theta\right)}{\cos\theta \times \left(1-2\sin^2\theta\right)}$$

$$= \tan\theta = \text{R.H.S}$$

(viii) $\left(\sin A + \operatorname{cosec} A\right)^2 + \left(\cos A + \sec A\right)^2 = 7 + \tan^2 A + \cot^2 A$

$$\text{L.H.S} = \left(\sin A + \operatorname{cosec} A\right)^2 + \left(\cos A + \sec A\right)^2$$

$$= \sin^2 A + \operatorname{cosec}^2 A + 2\sin A \operatorname{cosec} A + \cos^2 A + \sec^2 A + 2\cos A \sec A$$

$$= \left(\sin^2 A + \cos^2 A\right) + \left(\operatorname{cosec}^2 A + \sec^2 A\right) + 2\sin A\left(\frac{1}{\sin A}\right) + 2\cos A\left(\frac{1}{\cos A}\right)$$

$$= (1) + \left(1 + \cot^2 A + 1 + \tan^2 A\right) + (2) + (2)$$

$$= 7 + \tan^2 A + \cot^2 A$$

$$= \text{R.H.S}$$

(ix) $(\text{cosec}A - \sin A)(\sec A - \cos A) = \dfrac{1}{\tan A + \cot A}$

$$\text{L.H.S} = (\text{cosec}A - \sin A)(\sec A - \cos A)$$
$$= \left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right)$$
$$= \left(\frac{1-\sin^2 A}{\sin A}\right)\left(\frac{1-\cos^2 A}{\cos A}\right)$$
$$= \frac{(\cos^2 A)(\sin^2 A)}{\sin A \cos A}$$
$$= \sin A \cos A$$

$$\text{R.H.S} = \frac{1}{\tan A + \cot A}$$
$$= \frac{1}{\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}} = \frac{1}{\frac{\sin^2 A + \cos^2 A}{\sin A \cos A}}$$
$$= \frac{\sin A \cos A}{\sin^2 A + \cos^2 A} = \sin A \cos A$$

Hence, L.H.S = R.H.S

(x) $\left(\dfrac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\dfrac{1-\tan A}{1-\cot A}\right)^2 = \tan^2 A$

$$\frac{1+\tan^2 A}{1+\cot^2 A} = \frac{1+\frac{\sin^2 A}{\cos^2 A}}{1+\frac{\cos^2 A}{\sin^2 A}} = \frac{\frac{\cos^2 A + \sin^2 A}{\cos^2 A}}{\frac{\sin^2 A + \cos^2 A}{\sin^2 A}}$$
$$= \frac{\frac{1}{\cos^2 A}}{\frac{1}{\sin^2 A}} = \frac{\sin^2 A}{\cos^2 A}$$
$$= \tan^2 A$$

$$\left(\frac{1-\tan A}{1-\cot A}\right)^2 = \frac{1+\tan^2 A - 2\tan A}{1+\cot^2 A - 2\cot A}$$

$$= \frac{\sec^2 A - 2\tan A}{\operatorname{cosec}^2 A - 2\cot A}$$

$$= \frac{\dfrac{1}{\cos^2 A} - \dfrac{2\sin A}{\cos A}}{\dfrac{1}{\sin^2 A} - \dfrac{2\cos A}{\sin A}} = \frac{\dfrac{1-2\sin A\cos A}{\cos^2 A}}{\dfrac{1-2\sin A\cos A}{\sin^2 A}}$$

$$= \frac{\sin^2 A}{\cos^2 A} = \tan^2 A$$