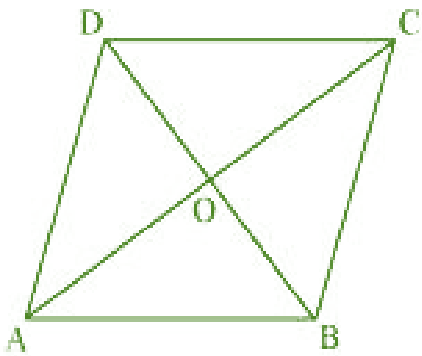
Miscellaneous Exercise on Chapter 12

Question 1:

Three vertices of a parallelogram ABCD are A (3, −1, 2), B (1, 2, −4) and C (−1, 1, 2). Find the coordinates of the fourth vertex.

Answer 1:

The three vertices of a parallelogram ABCD are given as A (3, −1, 2), B (1, 2, −4), and C (−1, 1, 2). Let the coordinates of the fourth vertex be D (x, y, z).



We know that the diagonals of a parallelogram bisect each other.

Therefore, in parallelogram ABCD, AC and BD bisect each other.

∴ Mid-point of AC = Mid-point of BD

$$\Rightarrow \left(\frac{3-1}{2}, \frac{-1+1}{2}, \frac{2+2}{2}\right) = \left(\frac{x+1}{2}, \frac{y+2}{2}, \frac{z-4}{2}\right)$$

$$\Rightarrow (1, 0, 2) = \left(\frac{x+1}{2}, \frac{y+2}{2}, \frac{z-4}{2}\right)$$

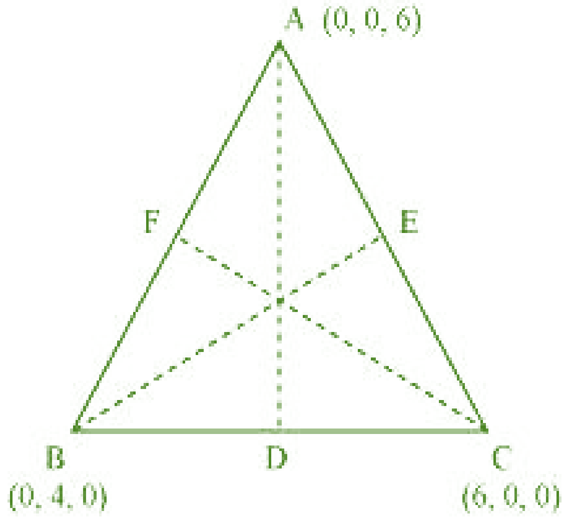
$$\Rightarrow \frac{x+1}{2} = 1, \frac{y+2}{2} = 0, \text{ and } \frac{z-4}{2} = 2$$

⇒ $x = 1$, $y = -2$, and $z = 8$

Thus, the coordinates of the fourth vertex are (1, −2, 8).

Question 2:
Find the lengths of the medians of the triangle with vertices A (0, 0, 6), B (0, 4, 0) and (6, 0, 0).

Answer 2:
Let AD, BE, and CF be the medians of the given triangle ABC.



Since AD is the median, D is the mid-point of BC.

$\therefore$Coordinates of point D = $\left(\frac{0+6}{2}, \frac{4+0}{2}, \frac{0+0}{2}\right)$ = (3, 2, 0)

$$AD = \sqrt{(0-3)^2 + (0-2)^2 + (6-0)^2} = \sqrt{9+4+36} = \sqrt{49} = 7$$

Since BE is the median, E is the mid-point of AC.

$$\therefore \text{Coordinates of point E} = \left(\frac{0+6}{2}, \frac{0+0}{2}, \frac{6+0}{2}\right) = (3,0,3)$$

$$BE = \sqrt{(3-0)^2 + (0-4)^2 + (3-0)^2} = \sqrt{9+16+9} = \sqrt{34}$$

Since CF is the median, F is the mid-point of AB.

$$\therefore \text{Coordinates of point F} = \left(\frac{0+0}{2}, \frac{0+4}{2}, \frac{6+0}{2}\right) = (0,2,3)$$

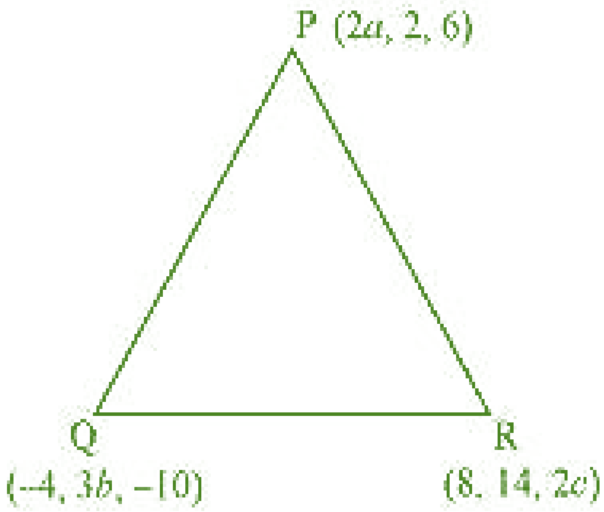
$$\text{Length of CF} = \sqrt{(6-0)^2 + (0-2)^2 + (0-3)^2} = \sqrt{36+4+9} = \sqrt{49} = 7$$

Thus, the lengths of the medians of ΔABC are $7, \sqrt{34}$, and 7.

Question 3:

If the origin is the centroid of the triangle PQR with vertices P (2*a*, 2, 6), Q (−4, 3*b*, −10) and R (8, 14, 2*c*), then find the values of *a*, *b* and *c*.

Answer 3:



It is known that the coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.

Therefore, coordinates of the centroid of ΔPQR

$$=\left(\frac{2a-4+8}{3}, \frac{2+3b+14}{3}, \frac{6-10+2c}{3}\right)=\left(\frac{2a+4}{3}, \frac{3b+16}{3}, \frac{2c-4}{3}\right)$$

It is given that origin is the centroid of ΔPQR.

$$\therefore (0,0,0)=\left(\frac{2a+4}{3}, \frac{3b+16}{3}, \frac{2c-4}{3}\right)$$

$$\Rightarrow \frac{2a+4}{3}=0, \frac{3b+16}{3}=0 \text{ and } \frac{2c-4}{3}=0$$

$$\Rightarrow a=-2, b=-\frac{16}{3} \text{ and } c=2$$

Thus, the respective values of *a*, *b*, and *c* are $-2, -\frac{16}{3}$, and 2.

Question 4:

Find the coordinates of a point on y-axis which are at a distance of $5\sqrt{2}$ from the point P (3, –2, 5).

Answer 4:

If a point is on the y-axis, then x-coordinate and the z-coordinate of the point are zero.
Let A (0, b, 0) be the point on the y-axis at a distance of $5\sqrt{2}$ from point P (3, –2, 5).

Accordingly, AP = $5\sqrt{2}$

$$\therefore \text{AP}^2 = 50$$
$$\Rightarrow (3-0)^2 + (-2-b)^2 + (5-0)^2 = 50$$
$$\Rightarrow 9 + 4 + b^2 + 4b + 25 = 50$$
$$\Rightarrow b^2 + 4b - 12 = 0$$
$$\Rightarrow b^2 + 6b - 2b - 12 = 0$$
$$\Rightarrow (b+6)(b-2) = 0$$
$$\Rightarrow b = -6 \text{ or } 2$$

Thus, the coordinates of the required points are (0, 2, 0) and (0, –6, 0).

Question 5:

A point R with x-coordinate 4 lies on the line segment joining the pointsP (2, –3, 4) and Q (8, 0, 10). Find the coordinates of the point R.

[**Hint** suppose R divides PQ in the ratio k: 1. The coordinates of the point R are given by

$$\left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right)]$$

Answer 5:

The coordinates of points P and Q are given as P (2, –3, 4) and Q (8, 0, 10).
Let R divide line segment PQ in the ratio k:1.

Hence, by section formula, the coordinates of point R are given by

$$\left(\frac{k(8)+2}{k+1}, \frac{k(0)-3}{k+1}, \frac{k(10)+4}{k+1}\right) = \left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right)$$

It is given that the x-coordinate of point R is 4.

$$\therefore \frac{8k+2}{k+1}=4$$
$$\Rightarrow 8k+2=4k+4$$
$$\Rightarrow 4k=2$$
$$\Rightarrow k=\frac{1}{2}$$

Therefore, the coordinates of point R are $\left(4, \frac{-3}{\frac{1}{2}+1}, \frac{10\left(\frac{1}{2}\right)+4}{\frac{1}{2}+1}\right)=(4,-2,6)$

Question 6:

If A and B be the points (3, 4, 5) and (–1, 3, –7), respectively, find the equation of the set of points P such that $PA^2 + PB^2 = k^2$, where k is a constant.

Answer 6:

The coordinates of points A and B are given as (3, 4, 5) and (–1, 3, –7) respectively. Let the coordinates of point P be (x, y, z).

On using distance formula, we obtain

$$\begin{aligned} PA^2 &= (x-3)^2+(y-4)^2+(z-5)^2 \\ &= x^2+9-6x+y^2+16-8y+z^2+25-10z \\ &= x^2-6x+y^2-8y+z^2-10z+50 \end{aligned}$$

$$\begin{aligned} PB^2 &= (x+1)^2+(y-3)^2+(z+7)^2 \\ &= x^2+2x+y^2-6y+z^2+14z+59 \end{aligned}$$

Now, if $PA^2 + PB^2 = k^2$, then

$$\left(x^2-6x+y^2-8y+z^2-10z+50\right)+\left(x^2+2x+y^2-6y+z^2+14z+59\right)=k^2$$
$$\Rightarrow 2x^2+2y^2+2z^2-4x-14y+4z+109=k^2$$
$$\Rightarrow 2\left(x^2+y^2+z^2-2x-7y+2z\right)=k^2-109$$
$$\Rightarrow x^2+y^2+z^2-2x-7y+2z=\frac{k^2-109}{2}$$

Thus, the required equation is $x^2+y^2+z^2-2x-7y+2z=\frac{k^2-109}{2}$