Mathematics

Exercise 13.2

Question 1:
Find the derivative of $x^2 - 2$ at $x = 10$.
Answer 1:
Let $f(x) = x^2 - 2$. Accordingly,

$$f'(10) = \lim_{h \to 0} \frac{f(10+h) - f(10)}{h}$$

$$= \lim_{h \to 0} \frac{\left[(10+h)^2 - 2\right] - (10^2 - 2)}{h}$$

$$= \lim_{h \to 0} \frac{10^2 + 2.10.h + h^2 - 2 - 10^2 + 2}{h}$$

$$= \lim_{h \to 0} \frac{20h + h^2}{h}$$

$$= \lim_{h \to 0} (20 + h) = (20 + 0) = 20$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20.

Question 2:
Find the derivative of $99x$ at $x = 100$.
Answer 2:
Let $f(x) = 99x$. Accordingly,

$$f'(100) = \lim_{h \to 0} \frac{f(100+h) - f(100)}{h}$$

$$= \lim_{h \to 0} \frac{99(100+h) - 99(100)}{h}$$

$$= \lim_{h \to 0} \frac{99 \times 100 + 99h - 99 \times 100}{h}$$

$$= \lim_{h \to 0} \frac{99h}{h}$$

$$= \lim_{h \to 0} (99) = 99$$

Thus, the derivative of $99x$ at $x = 100$ is 99.

Question 3:

Find the derivative of x at $x = 1$.

Answer 3:

Let $f(x) = x$. Accordingly,

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h) - 1}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} (1)$$

$$= 1$$

Thus, the derivative of x at $x = 1$ is 1.

Question 4:

Find the derivative of the following functions from first principle.

(i) $x^3 - 27$ (ii) $(x - 1)(x - 2)$

(iii) $\dfrac{1}{x^2}$ (iv) $\dfrac{x+1}{x-1}$

Answer 4:

(i) Let $f(x) = x^3 - 27$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[(x+h)^3 - 27\right] - (x^3 - 27)}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{h^3 + 3x^2h + 3xh^2}{h}$$

$$= \lim_{h \to 0} \left(h^2 + 3x^2 + 3xh\right)$$

$$= 0 + 3x^2 + 0 = 3x^2$$

(ii) Let $f(x) = (x - 1)(x - 2)$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h}$$

$$= \lim_{h \to 0} \frac{\left(x^2 + hx - 2x + hx + h^2 - 2h - x - h + 2\right) - \left(x^2 - 2x - x + 2\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(hx + hx + h^2 - 2h - h\right)}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2 - 3h}{h}$$

$$= \lim_{h \to 0} (2x + h - 3)$$

$$= (2x + 0 - 3)$$

$$= 2x - 3$$

(iii) Let $f(x) = \dfrac{1}{x^2}$

Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x^2 - x^2 - h^2 - 2hx}{x^2(x+h)^2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-h^2 - 2hx}{x^2(x+h)^2}\right]$$

$$= \lim_{h \to 0}\left[\frac{-h - 2x}{x^2(x+h)^2}\right]$$

$$= \frac{0 - 2x}{x^2(x+0)^2} = \frac{-2}{x^3}$$

(iv) Let $f(x) = \dfrac{x+1}{x-1}$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\dfrac{x+h+1}{x+h-1} - \dfrac{x+1}{x-1}\right)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{(x^2 + hx + x - x - h - 1) - (x^2 + hx - x + x + h - 1)}{(x-1)(x+h-1)} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{-2h}{(x-1)(x+h-1)} \right]$$

$$= \lim_{h \to 0} \left[\frac{-2}{(x-1)(x+h-1)} \right]$$

$$= \frac{-2}{(x-1)(x-1)} = \frac{-2}{(x-1)^2}$$

Question 5:

For the function

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \ldots + \frac{x^2}{2} + x + 1$$

Prove that $f'(1) = 100 f'(0)$

Answer 5:
The given function is

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \ldots + \frac{x^2}{2} + x + 1$$

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left[\frac{x^{100}}{100} + \frac{x^{99}}{99} + \ldots + \frac{x^2}{2} + x + 1\right]$$

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{x^{100}}{100}\right) + \frac{d}{dx}\left(\frac{x^{99}}{99}\right) + \ldots + \frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{dx}(x) + \frac{d}{dx}(1)$$

On using theorem $\dfrac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$$\frac{d}{dx} f(x) = \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + \ldots + \frac{2x}{2} + 1 + 0$$

$$= x^{99} + x^{98} + \ldots + x + 1$$

$$\therefore f'(x) = x^{99} + x^{98} + \ldots + x + 1$$

At $x = 0$,

$$f'(0) = 1$$

At $x = 1$,

$$f'(1) = 1^{99} + 1^{98} + \ldots + 1 + 1 = [1 + 1 + \ldots + 1 + 1]_{100 \text{ terms}} = 1 \times 100 = 100$$

Thus, $f'(1) = 100 \times f^1(0)$

Question 6:

Find the derivative of $\quad x^n + ax^{n-1} + a^2 x^{n-2} + \ldots + a^{n-1}x + a^n \quad$ for some fixed real number a.

Answer 6:

Let $f(x) = x^n + ax^{n-1} + a^2 x^{n-2} + \ldots + a^{n-1} x + a^n$

$$\therefore f'(x) = \frac{d}{dx}\left(x^n + ax^{n-1} + a^2 x^{n-2} + \ldots + a^{n-1} x + a^n\right)$$

$$= \frac{d}{dx}\left(x^n\right) + a\frac{d}{dx}\left(x^{n-1}\right) + a^2 \frac{d}{dx}\left(x^{n-2}\right) + \ldots + a^{n-1}\frac{d}{dx}(x) + a^n \frac{d}{dx}(1)$$

On using theorem $\dfrac{d}{dx} x^n = nx^{n-1}$, we obtain

$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \ldots + a^{n-1} + a^n(0)$$

$$= nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \ldots + a^{n-1}$$

Question 7:

For some constants a and b, find the derivative of

(i) $(x - a)(x - b)$ (ii) $(ax^2 + b)^2$ (iii) $\dfrac{x-a}{x-b}$

Answer 7:

(i) Let $f(x) = (x - a)(x - b)$

$$\Rightarrow f(x) = x^2 - (a+b)x + ab$$

$$\therefore f'(x) = \frac{d}{dx}\left(x^2 - (a+b)x + ab\right)$$

$$= \frac{d}{dx}\left(x^2\right) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$$

On using theorem $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$, we obtain

$$f'(x) = 2x - (a+b) + 0 = 2x - a - b$$

(ii) Let $f(x) = (ax^2 + b)^2$

$$\Rightarrow f(x) = a^2 x^4 + 2abx^2 + b^2$$

$$\therefore f'(x) = \frac{d}{dx}(a^2 x^4 + 2abx^2 + b^2) = a^2 \frac{d}{dx}(x^4) + 2ab \frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$$

On using theorem $\frac{d}{dx} x^n = nx^{n-1}$, we obtain

$$f'(x) = a^2(4x^3) + 2ab(2x) + b^2(0)$$

$$= 4a^2 x^3 + 4abx$$

$$= 4ax(ax^2 + b)$$

(iii) Let $f(x) = \frac{(x-a)}{(x-b)}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

By quotient rule,

$$f'(x) = \frac{(x-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{(x-b)(1) - (x-a)(1)}{(x-b)^2}$$

$$= \frac{x-b-x+a}{(x-b)^2}$$

$$= \frac{a-b}{(x-b)^2}$$

Question 8:

Find the derivative of $\dfrac{x^n - a^n}{x - a}$ for some constant a.

Answer 8:

$$\text{Let } f(x) = \frac{x^n - a^n}{x - a}$$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x^n - a^n}{x - a}\right)$$

By quotient rule,

$$f'(x) = \frac{(x-a)\dfrac{d}{dx}(x^n - a^n) - (x^n - a^n)\dfrac{d}{dx}(x-a)}{(x-a)^2}$$

$$= \frac{(x-a)(nx^{n-1} - 0) - (x^n - a^n)}{(x-a)^2}$$

$$= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$$

Question 9:

Find the derivative of

(i) $2x - \dfrac{3}{4}$

(ii) $(5x^3 + 3x - 1)(x - 1)$

(iii) $x^{-3}(5 + 3x)$

(iv) $x^5(3 - 6x^{-9})$

(v) $x^{-4}(3 - 4x^{-5})$

(vi) $\dfrac{2}{x+1} - \dfrac{x^2}{3x-1}$

Answer 9:

(i) Let $f(x) = 2x - \dfrac{3}{4}$

$$f'(x) = \frac{d}{dx}\left(2x - \frac{3}{4}\right)$$
$$= 2\frac{d}{dx}(x) - \frac{d}{dx}\left(\frac{3}{4}\right)$$
$$= 2 - 0$$
$$= 2$$

(ii) Let $f(x) = (5x^3 + 3x - 1)(x - 1)$

By Leibnitz product rule,

$$f'(x) = \left(5x^3 + 3x - 1\right)\frac{d}{dx}(x-1) + (x-1)\frac{d}{dx}\left(5x^3 + 3x - 1\right)$$
$$= \left(5x^3 + 3x - 1\right)(1) + (x-1)\left(5.3x^2 + 3 - 0\right)$$
$$= \left(5x^3 + 3x - 1\right) + (x-1)\left(15x^2 + 3\right)$$
$$= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3$$
$$= 20x^3 - 15x^2 + 6x - 4$$

(iii) Let $f(x) = x^{-3}(5 + 3x)$

By Leibnitz product rule,

$$f'(x) = x^{-3} \frac{d}{dx}(5+3x) + (5+3x)\frac{d}{dx}(x^{-3})$$

$$= x^{-3}(0+3) + (5+3x)(-3x^{-3-1})$$

$$= x^{-3}(3) + (5+3x)(-3x^{-4})$$

$$= 3x^{-3} - 15x^{-4} - 9x^{-3}$$

$$= -6x^{-3} - 15x^{-4}$$

$$= -3x^{-3}\left(2 + \frac{5}{x}\right)$$

$$= \frac{-3x^{-3}}{x}(2x+5)$$

$$= \frac{-3}{x^4}(5+2x)$$

(iv) Let $f(x) = x^5(3 - 6x^{-9})$

By Leibnitz product rule,

$$f'(x) = x^5 \frac{d}{dx}(3 - 6x^{-9}) + (3 - 6x^{-9})\frac{d}{dx}(x^5)$$

$$= x^5\{0 - 6(-9)x^{-9-1}\} + (3 - 6x^{-9})(5x^4)$$

$$= x^5(54x^{-10}) + 15x^4 - 30x^{-5}$$

$$= 54x^{-5} + 15x^4 - 30x^{-5}$$

$$= 24x^{-5} + 15x^4$$

$$= 15x^4 + \frac{24}{x^5}$$

(v) Let $f(x) = x^{-4}(3 - 4x^{-5})$

By Leibnitz product rule,

$$f'(x) = x^{-4}\frac{d}{dx}\left(3 - 4x^{-5}\right) + \left(3 - 4x^{-5}\right)\frac{d}{dx}\left(x^{-4}\right)$$

$$= x^{-4}\left\{0 - 4(-5)x^{-5-1}\right\} + \left(3 - 4x^{-5}\right)(-4)x^{-4-1}$$

$$= x^{-4}\left(20x^{-6}\right) + \left(3 - 4x^{-5}\right)\left(-4x^{-5}\right)$$

$$= 20x^{-10} - 12x^{-5} + 16x^{-10}$$

$$= 36x^{-10} - 12x^{-5}$$

$$= -\frac{12}{x^5} + \frac{36}{x^{10}}$$

(vi) Let $f(x) = \dfrac{2}{x+1} - \dfrac{x^2}{3x-1}$

$$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1}\right) - \frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$$

By quotient rule,

$$f'(x) = \left[\frac{(x+1)\dfrac{d}{dx}(2) - 2\dfrac{d}{dx}(x+1)}{(x+1)^2}\right] - \left[\frac{(3x-1)\dfrac{d}{dx}(x^2) - x^2\dfrac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2}\right] - \left[\frac{(3x-1)(2x) - (x^2)(3)}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{3x^2 - 2x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

Question 10:

Find the derivative of cos x from first principle.

Answer 10:

Let $f(x) = \cos x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{-\cos x (1 - \cos h) - \sin x \sin h}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{-\cos x (1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right]$$

$$= -\cos x \left(\lim_{h \to 0} \frac{1 - \cos h}{h} \right) - \sin x \lim_{h \to 0} \left(\frac{\sin h}{h} \right)$$

$$= -\cos x (0) - \sin x (1) \qquad \left[\lim_{h \to 0} \frac{1 - \cos h}{h} = 0 \text{ and } \lim_{h \to 0} \frac{\sin h}{h} = 1 \right]$$

$$= -\sin x$$

$$\therefore f'(x) = -\sin x$$

Question 11:

Find the derivative of the following functions:

(i) sin x cos x (ii) sec x (iii) 5 sec x + 4 cos x

(iv) cosec x (v) 3cot x + 5cosec x (vi) 5sin x – 6cos x + 7

(vii) 2tan x – 7sec x

Answer 11:

(i)   Let $f(x) = \sin x \cos x$.

Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h)\cos(x+h) - \sin x \cos x}{h}$$

$$= \lim_{h \to 0} \frac{1}{2h}\left[2\sin(x+h)\cos(x+h) - 2\sin x \cos x\right]$$

$$= \lim_{h \to 0} \frac{1}{2h}\left[\sin 2(x+h) - \sin 2x\right]$$

$$= \lim_{h \to 0} \frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2} \cdot \sin\frac{2x+2h-2x}{2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\cos\frac{4x+2h}{2}\sin\frac{2h}{2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\cos(2x+h)\sin h\right]$$

$$= \lim_{h \to 0}\cos(2x+h).\lim_{h \to 0}\frac{\sin h}{h}$$

$$= \cos(2x+0).1$$

$$= \cos 2x$$

(ii)   Let $f(x) = \sec x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right]$$

$$= \frac{1}{\cos x} . \lim_{h \to 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right]$$

$$= \frac{1}{\cos x} . \lim_{h \to 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos(x+h)} \right]$$

$$= \frac{1}{\cos x} . \lim_{h \to 0} \frac{\left[\sin\left(\frac{2x+h}{2}\right) \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right]}{\cos(x+h)}$$

$$= \frac{1}{\cos x} . \lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} . \lim_{h \to 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}$$

$$= \frac{1}{\cos x} . 1 . \frac{\sin x}{\cos x}$$

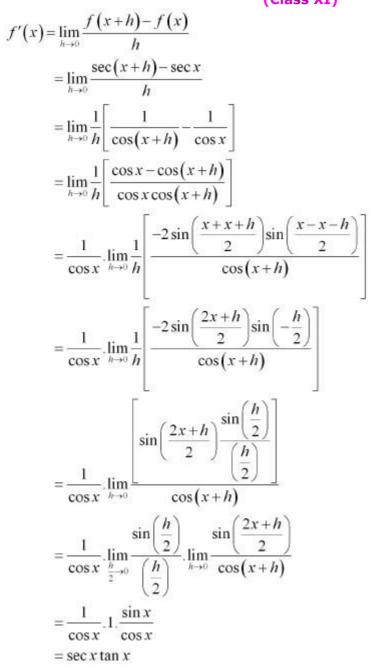$$= \sec x \tan x$$

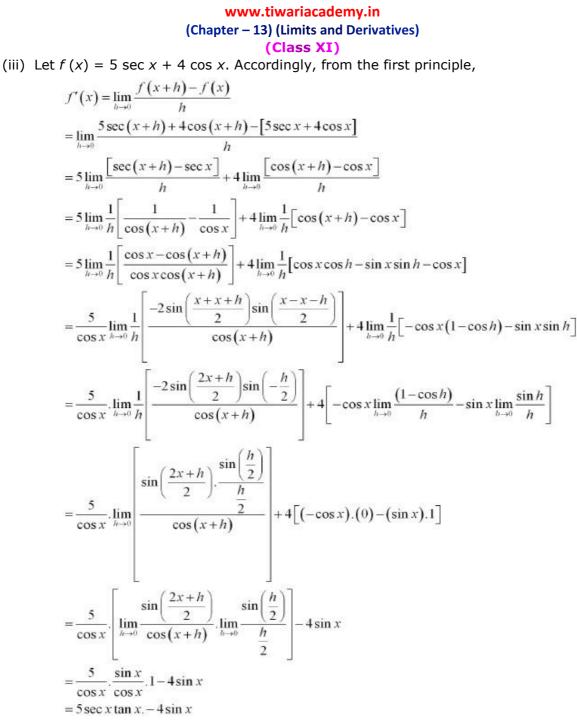(iii) Let $f(x) = 5\sec x + 4\cos x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{5\sec(x+h) + 4\cos(x+h) - [5\sec x + 4\cos x]}{h}$$

$$= 5\lim_{h\to 0} \frac{[\sec(x+h) - \sec x]}{h} + 4\lim_{h\to 0} \frac{[\cos(x+h) - \cos x]}{h}$$

$$= 5\lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right] + 4\lim_{h\to 0} \frac{1}{h}[\cos(x+h) - \cos x]$$

$$= 5\lim_{h\to 0} \frac{1}{h}\left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)}\right] + 4\lim_{h\to 0} \frac{1}{h}[\cos x \cos h - \sin x \sin h - \cos x]$$

$$= \frac{5}{\cos x}\lim_{h\to 0} \frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right] + 4\lim_{h\to 0} \frac{1}{h}\left[-\cos x(1-\cos h) - \sin x \sin h\right]$$

$$= \frac{5}{\cos x}\cdot\lim_{h\to 0} \frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right] + 4\left[-\cos x \lim_{h\to 0}\frac{(1-\cos h)}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$= \frac{5}{\cos x}\cdot\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right)\cdot\dfrac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}{\cos(x+h)}\right] + 4[(-\cos x)\cdot(0) - (\sin x)\cdot 1]$$

$$= \frac{5}{\cos x}\cdot\left[\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}\cdot\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right] - 4\sin x$$

$$= \frac{5}{\cos x}\cdot\frac{\sin x}{\cos x}\cdot 1 - 4\sin x$$

$$= 5\sec x \tan x. - 4\sin x$$

(iv) Let $f(x) = \operatorname{cosec} x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{h} \left[\operatorname{cosec}(x+h) - \operatorname{cosec} x \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin(x+h)\sin x} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{2\cos\left(\dfrac{x+x+h}{2}\right) \cdot \sin\left(\dfrac{x-x-h}{2}\right)}{\sin(x+h)\sin x} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{2\cos\left(\dfrac{2x+h}{2}\right)\sin\left(-\dfrac{h}{2}\right)}{\sin(x+h)\sin x} \right]$$

$$= \lim_{h \to 0} \frac{-\cos\left(\dfrac{2x+h}{2}\right) \cdot \dfrac{\sin\left(\dfrac{h}{2}\right)}{\left(\dfrac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$= \lim_{h \to 0} \left(\frac{-\cos\left(\dfrac{2x+h}{2}\right)}{\sin(x+h)\sin x} \right) \cdot \lim_{\frac{h}{2} \to 0} \frac{\sin\left(\dfrac{h}{2}\right)}{\left(\dfrac{h}{2}\right)}$$

$$= \left(\frac{-\cos x}{\sin x \sin x} \right) \cdot 1$$

$$= -\operatorname{cosec} x \cot x$$

(v) Let $f(x) = 3\cot x + 5\operatorname{cosec} x$.

Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3\cot(x+h) + 5\operatorname{cosec}(x+h) - 3\cot x - 5\operatorname{cosec} x}{h}$$

$$= 3\lim_{h \to 0} \frac{1}{h}\big[\cot(x+h) - \cot x\big] + 5\lim_{h \to 0} \frac{1}{h}\big[\operatorname{cosec}(x+h) - \operatorname{cosec} x\big] \qquad \ldots(1)$$

Now, $\lim_{h \to 0} \dfrac{1}{h}\big[\cot(x+h) - \cot x\big]$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\cos(x+h)\sin x - \cos x \sin(x+h)}{\sin x \sin(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x - x - h)}{\sin x \sin(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(-h)}{\sin x \sin(x+h)}\right]$$

$$= -\left(\lim_{h \to 0} \frac{\sin h}{h}\right)\cdot\left(\lim_{h \to 0} \frac{1}{\sin x \cdot \sin(x+h)}\right)$$

$$= -1.\frac{1}{\sin x \cdot \sin(x+0)} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x \qquad \ldots(2)$$

$$\lim_{h \to 0} \frac{1}{h} \left[\operatorname{cosec}(x+h) - \operatorname{cosec} x \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin(x+h)\sin x} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{2\cos\left(\dfrac{x+x+h}{2}\right) \cdot \sin\left(\dfrac{x-x-h}{2}\right)}{\sin(x+h)\sin x} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{2\cos\left(\dfrac{2x+h}{2}\right)\sin\left(-\dfrac{h}{2}\right)}{\sin(x+h)\sin x} \right]$$

$$= \lim_{h \to 0} \frac{-\cos\left(\dfrac{2x+h}{2}\right) \cdot \dfrac{\sin\left(\dfrac{h}{2}\right)}{\left(\dfrac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$= \lim_{h \to 0} \left(\frac{-\cos\left(\dfrac{2x+h}{2}\right)}{\sin(x+h)\sin x} \right) \cdot \lim_{\frac{h}{2} \to 0} \frac{\sin\left(\dfrac{h}{2}\right)}{\left(\dfrac{h}{2}\right)}$$

$$= \left(\frac{-\cos x}{\sin x \sin x} \right) \cdot 1$$

$$= -\operatorname{cosec} x \cot x \qquad \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$f'(x) = -3\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x$$

(vi) Let $f(x) = 5\sin x - 6\cos x + 7$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[5\sin(x+h) - 6\cos(x+h) + 7 - 5\sin x + 6\cos x - 7\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[5\{\sin(x+h) - \sin x\} - 6\{\cos(x+h) - \cos x\}\right]$$

$$= 5\lim_{h \to 0} \frac{1}{h}\left[\sin(x+h) - \sin x\right] - 6\lim_{h \to 0} \frac{1}{h}\left[\cos(x+h) - \cos x\right]$$

$$= 5\lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right] - 6\lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= 5\lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}\right] - 6\lim_{h \to 0}\left[\frac{-\cos x(1-\cos h) - \sin x \sin h}{h}\right]$$

$$= 5\lim_{h \to 0}\left(\cos\left(\frac{2x+h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right) - 6\lim_{h \to 0}\left[\frac{-\cos x(1-\cos h)}{h} - \frac{\sin x \sin h}{h}\right]$$

$$= 5\left[\lim_{h \to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{\frac{h}{2} \to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right] - 6\left[(-\cos x)\left(\lim_{h \to 0}\frac{1-\cos h}{h}\right) - \sin x \lim_{h \to 0}\left(\frac{\sin h}{h}\right)\right]$$

$$= 5\cos x.1 - 6\left[(-\cos x).(0) - \sin x.1\right]$$

$$= 5\cos x + 6\sin x$$

(vii) Let $f(x) = 2\tan x - 7\sec x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[2 \tan(x+h) - 7 \sec(x+h) - 2 \tan x + 7 \sec x \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[2\{\tan(x+h) - \tan x\} - 7\{\sec(x+h) - \sec x\} \right]$$

$$= 2 \lim_{h \to 0} \frac{1}{h} \left[\tan(x+h) - \tan x \right] - 7 \lim_{h \to 0} \frac{1}{h} \left[\sec(x+h) - \sec x \right]$$

$$= 2 \lim_{h \to 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] - 7 \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right]$$

$$= 2 \lim_{h \to 0} \frac{1}{h} \left[\frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \to 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right]$$

$$= 2 \lim_{h \to 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \to 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos x \cos(x+h)} \right]$$

$$= 2 \lim_{h \to 0} \left[\left(\frac{\sin h}{h}\right) \frac{1}{\cos x \cos(x+h)} \right] - 7 \lim_{h \to 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos x \cos(x+h)} \right]$$

$$= 2 \left(\lim_{h \to 0} \frac{\sin h}{h} \right) \left(\lim_{h \to 0} \frac{1}{\cos x \cos(x+h)} \right) - 7 \left(\lim_{\frac{h}{2} \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \right) \left(\lim_{h \to 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x \cos(x+h)} \right)$$

$$= 2.1. \frac{1}{\cos x \cos x} - 7.1 \left(\frac{\sin x}{\cos x \cos x} \right)$$

$$= 2 \sec^2 x - 7 \sec x \tan x$$