Mathematics

Exercise 4.1

Question 1:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1+3+3^2+...+3^{n-1}=\frac{(3^n-1)}{2}$$

Answer 1:

Let the given statement be P(n), i.e.,

P(n): $1 + 3 + 3^2 + ...+ 3^{n-1} = \frac{(3^n-1)}{2}$

For $n = 1$, we have

P(1):= $\frac{(3^1-1)}{2}=\frac{3-1}{2}=\frac{2}{2}=1$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$1+3+3^2+...+3^{k-1}=\frac{(3^k-1)}{2} \quad ...(i)$$

We shall now prove that P(k + 1) is true.

Consider

$1 + 3 + 3^2 + ... + 3^{k-1} + 3^{(k+1)-1}$

$= (1 + 3 + 3^2 +... + 3^{k-1}) + 3^k$

$$= \frac{(3^k - 1)}{2} + 3^k \quad \text{[Using (i)]}$$
$$= \frac{(3^k - 1) + 2.3^k}{2}$$
$$= \frac{(1+2)3^k - 1}{2}$$
$$= \frac{3.3^k - 1}{2}$$
$$= \frac{3^{k+1} - 1}{2}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 2:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Answer 2:

Let the given statement be P(n), i.e.,

P(n): $1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

For n = 1, we have

P(1): $1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{1.2}{2}\right)^2 = 1^2 = 1$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$1^3+2^3+3^3+....+k^3=\left(\frac{k(k+1)}{2}\right)^2 \quad ...(i)$$

We shall now prove that P(k + 1) is true.
Consider

$1^3 + 2^3 + 3^3 + ... + k^3 + (k + 1)^3$
$= (1^3 + 2^3 + 3^3 + ... + k^3) + (k + 1)^3$

$$=\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 \quad [\text{Using (i)}]$$

$$=\frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$=\frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$=\frac{(k+1)^2\{k^2+4(k+1)\}}{4}$$

$$=\frac{(k+1)^2\{k^2+4k+4\}}{4}$$

$$=\frac{(k+1)^2(k+2)^2}{4}$$

$$=\frac{(k+1)^2(k+1+1)^2}{4}$$

$$=\left(\frac{(k+1)(k+1+1)}{2}\right)^2$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 3:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+...+\frac{1}{(1+2+3+...n)}=\frac{2n}{(n+1)}$$

Answer 3:

Let the given statement be P(n), i.e.,

$$P(n):\quad 1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...n}=\frac{2n}{n+1}$$

For n = 1, we have

P(1): 1 = $\frac{2.1}{1+1}=\frac{2}{2}=1$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$1+\frac{1}{1+2}+...+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+k}=\frac{2k}{k+1} \qquad ... \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k}+\frac{1}{1+2+3+\ldots+k+(k+1)}$$

$$=\left(1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots k}\right)+\frac{1}{1+2+3+\ldots+k+(k+1)}$$

$$=\frac{2k}{k+1}+\frac{1}{1+2+3+\ldots+k+(k+1)} \quad \text{[Using (i)]}$$

$$=\frac{2k}{k+1}+\frac{1}{\left(\frac{(k+1)(k+1+1)}{2}\right)} \quad \left[1+2+3+\ldots+n=\frac{n(n+1)}{2}\right]$$

$$=\frac{2k}{(k+1)}+\frac{2}{(k+1)(k+2)}$$

$$=\frac{2}{(k+1)}\left(k+\frac{1}{k+2}\right)$$

$$=\frac{2}{(k+1)}\left(\frac{k^2+2k+1}{k+2}\right)$$

$$=\frac{2}{(k+1)}\left[\frac{(k+1)^2}{k+2}\right]$$

$$=\frac{2(k+1)}{(k+2)}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 4:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$1.2.3 + 2.3.4 + \ldots + n(n + 1)(n + 2) = \frac{n(n+1)(n+2)(n+3)}{4}$

Answer 4:

Let the given statement be P(n), i.e.,

P(n): $1.2.3 + 2.3.4 + \ldots + n(n + 1)(n + 2) = \frac{n(n+1)(n+2)(n+3)}{4}$

For $n = 1$, we have

P(1): $1.2.3 = 6 = \frac{1(1+1)(1+2)(1+3)}{4} = \frac{1.2.3.4}{4} = 6$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$1.2.3 + 2.3.4 + \ldots + k(k + 1)(k + 2) = \frac{k(k+1)(k+2)(k+3)}{4}$ ... (i)

We shall now prove that P(k + 1) is true.

Consider

$1.2.3 + 2.3.4 + \ldots + k(k + 1)(k + 2) + (k + 1)(k + 2)(k + 3)$

$= \{1.2.3 + 2.3.4 + \ldots + k(k + 1)(k + 2)\} + (k + 1)(k + 2)(k + 3)$

$$=\frac{k(k+1)(k+2)(k+3)}{4}+(k+1)(k+2)(k+3) \qquad [\text{Using (i)}]$$

$$=(k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right)$$

$$=\frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$$=\frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}$$

Thus, P(*k* + 1) is true whenever P(*k*) is true.

Hence, by the principle of mathematical induction, statement P(*n*) is true for all natural numbers i.e., N.

Question 5:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.3+2.3^2+3.3^3+\ldots+n.3^n=\frac{(2n-1)3^{n+1}+3}{4}$$

Answer 5:

Let the given statement be P(*n*), i.e.,

P(*n*) : $1.3+2.3^2+3.3^3+\ldots+n3^n=\frac{(2n-1)3^{n+1}+3}{4}$

For *n* = 1, we have

P(1): 1.3 = 3 $=\frac{(2.1-1)3^{1+1}+3}{4}=\frac{3^2+3}{4}=\frac{12}{4}=3$, which is true.

Let P(*k*) be true for some positive integer *k*, i.e.,

$$1.3+2.3^2+3.3^3+...+k3^k=\frac{(2k-1)3^{k+1}+3}{4} \qquad ...(i)$$

We shall now prove that P(k + 1) is true.
Consider

$1.3 + 2.3^2 + 3.3^3 + ... + k.3^k + (k + 1).3^{k+1}$

$= (1.3 + 2.3^2 + 3.3^3 + ...+ k.3^k) + (k + 1).3^{k+1}$

$$=\frac{(2k-1)3^{k+1}+3}{4}+(k+1)3^{k+1} \qquad [\text{Using (i)}]$$
$$=\frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4}$$
$$=\frac{3^{k+1}\{2k-1+4(k+1)\}+3}{4}$$
$$=\frac{3^{k+1}\{6k+3\}+3}{4}$$
$$=\frac{3^{k+1}.3\{2k+1\}+3}{4}$$
$$=\frac{3^{(k+1)+1}\{2k+1\}+3}{4}$$
$$=\frac{\{2(k+1)-1\}3^{(k+1)+1}+3}{4}$$

Thus, P(k + 1) is true whenever P(k) is true.
Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 6:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.2+2.3+3.4+...+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$$

Answer 6:

Let the given statement be P(*n*), i.e.,

P(*n*): $1.2+2.3+3.4+...+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$

For *n* = 1, we have

P(1): $1.2=2=\frac{1(1+1)(1+2)}{3}=\frac{1.2.3}{3}=2$, which is true.

Let P(*k*) be true for some positive integer *k*, i.e.,

$$1.2+2.3+3.4+.....+k.(k+1)=\left[\frac{k(k+1)(k+2)}{3}\right] \qquad ...\text{(i)}$$

We shall now prove that P(*k* + 1) is true.

Consider

$1.2 + 2.3 + 3.4 + ... + k.(k + 1) + (k + 1).(k + 2)$

$= [1.2 + 2.3 + 3.4 + ... + k.(k + 1)] + (k + 1).(k + 2)$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \qquad \text{[Using (i)]}$$

$$= (k+1)(k+2)\left(\frac{k}{3}+1\right)$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$$= \frac{(k+1)(k+1+1)(k+1+2)}{3}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 7:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.3 + 3.5 + 5.7 + ... + (2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}$$

Answer 7:

Let the given statement be P(n), i.e.,

P(n): $1.3 + 3.5 + 5.7 + ... + (2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}$

For n = 1, we have

$$P(1): 1.3 = 3 = \frac{1(4.1^2 + 6.1 - 1)}{3} = \frac{4+6-1}{3} = \frac{9}{3} = 3$$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$1.3 + 3.5 + 5.7 + \ldots\ldots + (2k-1)(2k+1) = \frac{k(4k^2+6k-1)}{3} \quad \ldots \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

(1.3 + 3.5 + 5.7 + … + (2k – 1) (2k + 1) + {2(k + 1) – 1}{2(k + 1) + 1}

$$= \frac{k(4k^2+6k-1)}{3} + (2k+2-1)(2k+2+1) \qquad \text{[Using (i)]}$$

$$= \frac{k(4k^2+6k-1)}{3} + (2k+1)(2k+3)$$

$$= \frac{k(4k^2+6k-1)}{3} + (4k^2+8k+3)$$

$$= \frac{k(4k^2+6k-1)+3(4k^2+8k+3)}{3}$$

$$= \frac{4k^3+6k^2-k+12k^2+24k+9}{3}$$

$$= \frac{4k^3+18k^2+23k+9}{3}$$

$$= \frac{4k^3+14k^2+9k+4k^2+14k+9}{3}$$

$$= \frac{k(4k^2+14k+9)+1(4k^2+14k+9)}{3}$$

$$= \frac{(k+1)(4k^2+14k+9)}{3}$$

$$= \frac{(k+1)\{4k^2+8k+4+6k+6-1\}}{3}$$

$$= \frac{(k+1)\{4(k^2+2k+1)+6(k+1)-1\}}{3}$$

$$= \frac{(k+1)\{4(k+1)^2+6(k+1)-1\}}{3}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 8:

Prove the following by using the principle of mathematical induction for all $n \in N$: $1.2 +$
$2.2^2 + 3.2^2 + \ldots + n.2^n = (n - 1)\, 2^{n+1} + 2$

Answer 8:

Let the given statement be P(n), i.e.,

P(n): $1.2 + 2.2^2 + 3.2^2 + \ldots + n.2^n = (n - 1)\, 2^{n+1} + 2$

For $n = 1$, we have

P(1): $1.2 = 2 = (1 - 1)\, 2^{1+1} + 2 = 0 + 2 = 2$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$1.2 + 2.2^2 + 3.2^2 + \ldots + k.2^k = (k - 1)\, 2^{k+1} + 2$ … (i)

We shall now prove that P($k + 1$) is true.

Consider

$$\left\{1.2 + 2.2^2 + 3.2^3 + \ldots + k.2^k\right\} + (k+1)\cdot 2^{k+1}$$
$$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$$
$$= 2^{k+1}\left\{(k-1) + (k+1)\right\} + 2$$
$$= 2^{k+1}.2k + 2$$
$$= k.2^{(k+1)+1} + 2$$
$$= \left\{(k+1) - 1\right\}2^{(k+1)+1} + 2$$

Thus, P($k + 1$) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 9:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

Answer 9:

Let the given statement be P(n), i.e.,

P(n): $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

For n = 1, we have

P(1): $\frac{1}{2}=1-\frac{1}{2^1}=\frac{1}{2}$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots.+\frac{1}{2^k}=1-\frac{1}{2^k} \qquad \ldots \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots\dots+\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$$

$$=\left(1-\frac{1}{2^k}\right)+\frac{1}{2^{k+1}} \qquad \text{[Using (i)]}$$

$$=1-\frac{1}{2^k}+\frac{1}{2.2^k}$$

$$=1-\frac{1}{2^k}\left(1-\frac{1}{2}\right)$$

$$=1-\frac{1}{2^k}\left(\frac{1}{2}\right)$$

$$=1-\frac{1}{2^{k+1}}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 10:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\dots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

Answer 10:

Let the given statement be P(n), i.e.,

$$\text{P}(n): \quad \frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\dots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

For n = 1, we have

$P(1) = \frac{1}{2.5} = \frac{1}{10} = \frac{1}{6.1+4} = \frac{1}{10}$, which is true.

Let P(*k*) be true for some positive integer *k*, i.e.,

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + ... + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4} \qquad ... (i)$$

We shall now prove that P(*k* + 1) is true.

Consider

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + ... + \frac{1}{(3k-1)(3k+2)} + \frac{1}{\{3(k+1)-1\}\{3(k+1)+2\}}$$

$$= \frac{k}{6k+4} + \frac{1}{(3k+3-1)(3k+3+2)} \qquad [\text{Using (i)}]$$

$$= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{1}{(3k+2)}\left(\frac{k}{2} + \frac{1}{3k+5}\right)$$

$$= \frac{1}{(3k+2)}\left(\frac{k(3k+5)+2}{2(3k+5)}\right)$$

$$= \frac{1}{(3k+2)}\left(\frac{3k^2+5k+2}{2(3k+5)}\right)$$

$$= \frac{1}{(3k+2)}\left(\frac{(3k+2)(k+1)}{2(3k+5)}\right)$$

$$= \frac{(k+1)}{6k+10}$$

$$= \frac{(k+1)}{6(k+1)+4}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.