Exercise 4.1

Question 11:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

Answer 11:

Let the given statement be P(n), i.e.,

P(n): $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$

For n = 1, we have

$P(1): \frac{1}{1\cdot2\cdot3}=\frac{1\cdot(1+3)}{4(1+1)(1+2)}=\frac{1\cdot4}{4\cdot2\cdot3}=\frac{1}{1\cdot2\cdot3}$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\ldots+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)} \quad \ldots \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$$\left[\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\ldots\ldots+\frac{1}{k(k+1)(k+2)}\right]+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \qquad \text{[Using (i)]}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)}{4}+\frac{1}{k+3}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)^2+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\left(k^2+6k+9\right)+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+6k^2+9k+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+2k^2+k+4k^2+8k+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\left(k^2+2k+1\right)+4\left(k^2+2k+1\right)}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+1)^2+4(k+1)^2}{4(k+3)}\right\}$$

$$=\frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(k+1)\{(k+1)+3\}}{4\{(k+1)+1\}\{(k+1)+2\}}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 12:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

Answer 12:

Let the given statement be P(*n*), i.e.,

$$P(n): a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

For *n* = 1, we have

$P(1): a = \frac{a(r^1 - 1)}{(r - 1)} = a$, which is true.

Let P(*k*) be true for some positive integer *k*, i.e.,

$$a + ar + ar^2 + \ldots\ldots.. + ar^{k-1} = \frac{a(r^k - 1)}{r - 1} \qquad \ldots \text{(i)}$$

We shall now prove that P(*k* + 1) is true.

Consider

$$\left\{a+ar+ar^2+......+ar^{k-1}\right\}+ar^{(k+1)-1}$$

$$=\frac{a\left(r^k-1\right)}{r-1}+ar^k \qquad \left[\text{Using}\,(\text{i})\right]$$

$$=\frac{a\left(r^k-1\right)+ar^k\left(r-1\right)}{r-1}$$

$$=\frac{a\left(r^k-1\right)+ar^{k+1}-ar^k}{r-1}$$

$$=\frac{ar^k-a+ar^{k+1}-ar^k}{r-1}$$

$$=\frac{ar^{k+1}-a}{r-1}$$

$$=\frac{a\left(r^{k+1}-1\right)}{r-1}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 13:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$$

Answer 13:

Let the given statement be P(n), i.e.,

$$P(n): \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$$

For $n = 1$, we have

$$P(1): \left(1+\frac{3}{1}\right)=4=(1+1)^2=2^2=4, \text{ which is true.}$$

Let P(k) be true for some positive integer k, i.e.,

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)=(k+1)^2 \qquad \ldots (1)$$

We shall now prove that P(k + 1) is true.
Consider

$$\left[\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)\right]\left\{1+\frac{\{2(k+1)+1\}}{(k+1)^2}\right\}$$

$$=(k+1)^2\left(1+\frac{2(k+1)+1}{(k+1)^2}\right) \qquad [\text{Using}(1)]$$

$$=(k+1)^2\left[\frac{(k+1)^2+2(k+1)+1}{(k+1)^2}\right]$$

$$=(k+1)^2+2(k+1)+1$$

$$=\{(k+1)+1\}^2$$

Thus, P(k + 1) is true whenever P(k) is true.
Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 14:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1)$$

Answer 14:

Let the given statement be P(n), i.e.,

$$P(n):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1)$$

For n = 1, we have

$P(1):\left(1+\frac{1}{1}\right)=2=(1+1)$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$P(k):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)=(k+1) \qquad \ldots (1)$$

We shall now prove that P(k + 1) is true.

Consider

$$\left[\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$$

$$=(k+1)\left(1+\frac{1}{k+1}\right) \qquad [\text{Using } (1)]$$

$$=(k+1)\left(\frac{(k+1)+1}{(k+1)}\right)$$

$$=(k+1)+1$$

Thus, P(*k* + 1) is true whenever P(*k*) is true.

Hence, by the principle of mathematical induction, statement P(*n*) is true for all natural numbers i.e., *N*.

Question 15:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

Answer 15:

Let the given statement be P(*n*), i.e.,

$$P(n) = 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

For $n = 1$, we have

$$P(1) = 1^2 = 1 = \frac{1(2.1-1)(2.1+1)}{3} = \frac{1.1.3}{3} = 1, \text{ which is true.}$$

Let P(k) be true for some positive integer k, i.e.,

$$P(k) = 1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3} \quad \ldots (1)$$

We shall now prove that P(k + 1) is true.

Consider

$$\left\{1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2\right\} + \left\{2(k+1)-1\right\}^2$$

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+2-1)^2 \quad [\text{Using } (1)]$$

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$$

$$= \frac{(2k+1)\left\{k(2k-1) + 3(2k+1)\right\}}{3}$$

$$= \frac{(2k+1)\left\{2k^2 - k + 6k + 3\right\}}{3}$$

$$=\frac{(2k+1)\{2k^2+5k+3\}}{3}$$
$$=\frac{(2k+1)\{2k^2+2k+3k+3\}}{3}$$
$$=\frac{(2k+1)\{2k(k+1)+3(k+1)\}}{3}$$
$$=\frac{(2k+1)(k+1)(2k+3)}{3}$$
$$=\frac{(k+1)\{2(k+1)-1\}\{2(k+1)+1\}}{3}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 16:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

Answer 16:

Let the given statement be P(n), i.e.,

$$P(n): \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

For $n=1$, we have

$$P(1)=\frac{1}{1.4}=\frac{1}{3.1+1}=\frac{1}{4}=\frac{1}{1.4}$$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$P(k)=\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1} \quad ...(1)$$

We shall now prove that P(k + 1) is true.

Consider

$$\left\{\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}\right\}+\frac{1}{\{3(k+1)-2\}\{3(k+1)+1\}}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)} \qquad [\text{Using } (1)]$$

$$=\frac{1}{(3k+1)}\left\{k+\frac{1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{k(3k+4)+1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{3k^2+4k+1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{3k^2+3k+k+1}{(3k+4)}\right\}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$=\frac{(k+1)}{3(k+1)+1}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true

for all natural numbers i.e., N.

Question 17:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

Answer 17:

Let the given statement be P(n), i.e.,

$$P(n):\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

For n = 1, we have

$P(1):\frac{1}{3.5}=\frac{1}{3(2.1+3)}=\frac{1}{3.5}$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$P(k):\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)} \qquad ...(1)$$

We shall now prove that P(k + 1) is true. Consider

$$\left[\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2k+1)(2k+3)}\right]+\frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}$$

$$=\frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)} \qquad [\text{Using } (1)]$$

$$=\frac{1}{(2k+3)}\left[\frac{k}{3}+\frac{1}{(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{k(2k+5)+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k^2+5k+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k^2+2k+3k+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k(k+1)+3(k+1)}{3(2k+5)}\right]$$

$$=\frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$$

$$=\frac{(k+1)}{3\{2(k+1)+3\}}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 18:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1+2+3+\ldots+n<\frac{1}{8}(2n+1)^2$$

Answer 18:

Let the given statement be P(*n*), i.e.,

$$P(n): 1+2+3+\ldots+n<\frac{1}{8}(2n+1)^2$$

It can be noted that P(*n*) is true for *n* = 1 since

$$1<\frac{1}{8}(2.1+1)^2=\frac{9}{8}.$$

Let P(*k*) be true for some positive integer *k*, i.e.,

$$1+2+\ldots+k<\frac{1}{8}(2k+1)^2 \qquad \ldots (1)$$

We shall now prove that P(*k* + 1) is true whenever P(*k*) is true.

Consider

$$(1+2+...+k)+(k+1)<\frac{1}{8}(2k+1)^2+(k+1) \qquad [\text{Using}(1)]$$

$$<\frac{1}{8}\left\{(2k+1)^2+8(k+1)\right\}$$

$$<\frac{1}{8}\left\{4k^2+4k+1+8k+8\right\}$$

$$<\frac{1}{8}\left\{4k^2+12k+9\right\}$$

$$<\frac{1}{8}(2k+3)^2$$

$$<\frac{1}{8}\left\{2(k+1)+1\right\}^2$$

Hence, $(1+2+3+...+k)+(k+1)<\frac{1}{8}(2k+1)^2+(k+1)$

Thus, P(k + 1) is true whenever P(k) is true.
Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 19:
Prove the following by using the principle of mathematical induction for all $n \in N$:

n (n + 1) (n + 5) is a multiple of 3.

Answer 19:
Let the given statement be P(n), i.e.,
P(n): n (n + 1) (n + 5), which is a multiple of 3.
It can be noted that P(n) is true for n = 1 since 1 (1 + 1) (1 + 5) = 12, which is a multiple of 3.

Let P(k) be true for some positive integer k, i.e.,

k (k + 1) (k + 5) is a multiple of 3.

$\therefore k (k + 1) (k + 5) = 3m$, where $m \in \mathbf{N}$ … (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$$(k+1)\{(k+1)+1\}\{(k+1)+5\}$$
$$=(k+1)(k+2)\{(k+5)+1\}$$
$$=(k+1)(k+2)(k+5)+(k+1)(k+2)$$
$$=\{k(k+1)(k+5)+2(k+1)(k+5)\}+(k+1)(k+2)$$
$$=3m+(k+1)\{2(k+5)+(k+2)\}$$
$$=3m+(k+1)\{2k+10+k+2\}$$
$$=3m+(k+1)(3k+12)$$
$$=3m+3(k+1)(k+4)$$
$$=3\{m+(k+1)(k+4)\}=3\times q,\text{ where } q=\{m+(k+1)(k+4)\} \text{ is some natural number}$$

Therefore, $(k+1)\{(k+1)+1\}\{(k+1)+5\}$ is a multiple of 3.

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 20:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$10^{2n-1} + 1$ is divisible by 11.

Answer 20:

Let the given statement be P(n), i.e.,

P(n): $10^{2n-1} + 1$ is divisible by 11.

It can be observed that P(n) is true for $n = 1$

since P(1) = $10^{2.1-1} + 1 = 11$, which is divisible by 11.

Let P(k) be true for some positive integer k,

i.e., $10^{2k-1} + 1$ is divisible by 11.

$\therefore 10^{2k-1} + 1 = 11m$, where $m \in \mathbf{N}$ ... (1)

We shall now prove that P($k + 1$) is true whenever P(k) is true.

Consider

$10^{2(k+1)-1}+1$

$=10^{2k+2-1}+1$

$=10^{2k+1}+1$

$=10^2\left(10^{2k-1}+1-1\right)+1$

$=10^2\left(10^{2k-1}+1\right)-10^2+1$

$=10^2.11m-100+1 \qquad \left[\text{Using (1)}\right]$

$=100\times 11m-99$

$=11(100m-9)$

$=11r$, where $r=(100m-9)$ is some natural number

Therefore, $10^{2(k+1)-1}+1$ is divisible by 11.

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.