Exercise 4.1

Question 21:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$x^{2n} - y^{2n}$ is divisible by $x + y$.

Answer 21:

Let the given statement be P(n), i.e.,

P(n): $x^{2n} - y^{2n}$ is divisible by $x + y$.

It can be observed that P(n) is true for $n = 1$.

This is so because $x^{2 \times 1} - y^{2 \times 1} = x^2 - y^2 = (x + y)(x - y)$ is divisible by $(x + y)$.

Let P(k) be true for some positive integer k, i.e.,

$x^{2k} - y^{2k}$ is divisible by $x + y$.

$\therefore$ Let $x^{2k} - y^{2k} = m(x + y)$, where $m \in \mathbf{N}$ ... (1)

We shall now prove that P($k + 1$) is true whenever P(k) is true.

Consider

$x^{2(k+1)} - y^{2(k+1)}$
$= x^{2k} \cdot x^2 - y^{2k} \cdot y^2$
$= x^2 \left(x^{2k} - y^{2k} + y^{2k}\right) - y^{2k} \cdot y^2$
$= x^2 \left\{m(x+y) + y^{2k}\right\} - y^{2k} \cdot y^2 \qquad$ [Using (1)]
$= m(x+y)x^2 + y^{2k} \cdot x^2 - y^{2k} \cdot y^2$
$= m(x+y)x^2 + y^{2k}\left(x^2 - y^2\right)$
$= m(x+y)x^2 + y^{2k}(x+y)(x-y)$
$= (x+y)\left\{mx^2 + y^{2k}(x-y)\right\}$, which is a factor of $(x+y)$.

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 22:

Prove the following by using the principle of mathematical induction for all $n \in N$: $3^{2n+2} - 8n - 9$ is divisible by 8.

Answer 22:

Let the given statement be P(n), i.e.,

P(n): $3^{2n+2} - 8n - 9$ is divisible by 8.

It can be observed that P(n) is true for $n = 1$

since $3^{2 \times 1 + 2} - 8 \times 1 - 9 = 64$, which is divisible by 8.

Let P(*k*) be true for some positive integer

k, i.e., $3^{2k + 2} - 8k - 9$ is divisible by 8.

$\therefore 3^{2k + 2} - 8k - 9 = 8m$; where $m \in \mathbf{N}$ ... (1)

We shall now prove that P(*k* + 1) is true whenever P(*k*) is true.

Consider

$$3^{2(k+1)+2} - 8(k+1) - 9$$
$$= 3^{2k+2} \cdot 3^2 - 8k - 8 - 9$$
$$= 3^2 \left(3^{2k+2} - 8k - 9 + 8k + 9\right) - 8k - 17$$
$$= 3^2 \left(3^{2k+2} - 8k - 9\right) + 3^2 (8k + 9) - 8k - 17$$
$$= 9.8m + 9(8k + 9) - 8k - 17$$
$$= 9.8m + 72k + 81 - 8k - 17$$
$$= 9.8m + 64k + 64$$
$$= 8(9m + 8k + 8)$$
$$= 8r, \text{ where } r = (9m + 8k + 8) \text{ is a natural number}$$

Therefore, $3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8.

Thus, P(*k* + 1) is true whenever P(*k*) is true.

Hence, by the principle of mathematical induction, statement P(*n*) is true for all natural numbers i.e., *N*.

Question 23:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$41^n - 14^n$ is a multiple of 27.

Answer 23:

Let the given statement be P(n), i.e.,

P(n):$41^n - 14^n$ is a multiple of 27.

It can be observed that P(n) is true for $n = 1$

since $41^1 - 14^1 = 27$, which is a multiple of 27.

Let P(k) be true for some positive integer k, i.e.,

$41^k - 14^k$ is a multiple of 27

$\therefore 41^k - 14^k = 27m$, where $m \in \mathbf{N}$ ……………………………. (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$41^{k+1} - 14^{k+1}$
$= 41^k \cdot 41 - 14^k \cdot 14$
$= 41(41^k - 14^k + 14^k) - 14^k \cdot 14$
$= 41(41^k - 14^k) + 41.14^k - 14^k \cdot 14$
$= 41.27m + 14^k (41 - 14)$
$= 41.27m + 27.14^k$
$= 27(41m - 14^k)$
$= 27 \times r$, where $r = (41m - 14^k)$ is a natural number
Therefore, $41^{k+1} - 14^{k+1}$ is a multiple of 27.

Thus, P(k + 1) is true whenever P(k) is true.
Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 24:
Prove the following by using the principle of mathematical induction for all $n \in \mathrm{N}$:
$(2n + 7) < (n + 3)^2$

Answer 24:
Let the given statement be P(n), i.e.,
P(n): $(2n + 7) < (n + 3)^2$
It can be observed that P(n) is true for $n = 1$
since $2.1 + 7 = 9 < (1 + 3)^2 = 16$, which is true.
Let P(k) be true for some positive integer k, i.e.,

$(2k + 7) < (k + 3)^2$ … (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$$\{2(k+1)+7\} = (2k+7)+2$$

$$\therefore \{2(k+1)+7\} = (2k+7)+2 < (k+3)^2 + 2 \qquad [\text{using } (1)]$$

$$2(k+1)+7 < k^2+6k+9+2$$

$$2(k+1)+7 < k^2+6k+11$$

$$\text{Now, } k^2+6k+11 < k^2+8k+16$$

$$\therefore 2(k+1)+7 < (k+4)^2$$

$$2(k+1)+7 < \{(k+1)+3\}^2$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.