Mathematics

Exercise 8.2

Question 1:

Find the coefficient of x^5 in $(x + 3)^8$

Answer 1:

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r .$$

Assuming that x^5 occurs in the $(r + 1)^{th}$ term of the expansion $(x + 3)^8$, we obtain

$$T_{r+1} = {}^8C_r (x)^{8-r} (3)^r$$

Comparing the indices of x in x^5 and in T_{r+1},

we obtain $r = 3$

Thus, the coefficient of x^5 is $\quad {}^8C_3 (3)^3 = \dfrac{8!}{3!5!} \times 3^3 = \dfrac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2 \cdot 5!} \cdot 3^3 = 1512$

Question 2:

Find the coefficient of $a^5 b^7$ in $(a - 2b)^{12}$

Answer 2:

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r .$$

Assuming that $a^5 b^7$ occurs in the $(r + 1)^{th}$ term of the expansion $(a - 2b)^{12}$, we obtain

$$T_{r+1} = {}^{12}C_r (a)^{12-r} (-2b)^r = {}^{12}C_r (-2)^r (a)^{12-r} (b)^r$$

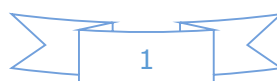Comparing the indices of a and b in $a^5 b^7$ and in T_{r+1},

we obtain $r = 7$

Thus, the coefficient of $a^5 b^7$ is

$${}^{12}C_7 (-2)^7 = -\dfrac{12!}{7!5!} \cdot 2^7 = -\dfrac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 7!} \cdot 2^7 = -(792)(128) = -101376$$

Question 3:

Write the general term in the expansion of $(x^2 - y)^6$

Answer 3:

It is known that the general term T_{r+1} {which is the $(r + 1)^{th}$ term} in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the general term in the expansion of $(x^2 - y^6)$ is

$$T_{r+1} = {}^6C_r \left(x^2\right)^{6-r} (-y)^r = (-1)^r \, {}^6C_r . x^{12-2r} . y^r$$

Question 4:

Write the general term in the expansion of $(x^2 - yx)^{12}$, $x \neq 0$

Answer 4:

It is known that the general term T_{r+1} {which is the $(r + 1)^{th}$ term} in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the general term in the expansion of $(x^2 - yx)^{12}$ is

$$T_{r+1} = {}^{12}C_r \left(x^2\right)^{12-r} (-yx)^r = (-1)^r \, {}^{12}C_r . x^{24-2r} . y^r . x^r = (-1)^r \, {}^{12}C_r . x^{24-r} . y^r$$

Question 5:

Find the 4th term in the expansion of $(x - 2y)^{12}$.

Answer 5:

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r .$$

Thus, the 4th term in the expansion of $(x - 2y)^{12}$ is

$$T_4 = T_{3+1} = {}^{12}C_3 (x)^{12-3} (-2y)^3 = (-1)^3 \cdot \frac{12!}{3!9!} \cdot x^9 \cdot (2)^3 \cdot y^3 = -\frac{12 \cdot 11 \cdot 10}{3 \cdot 2} \cdot (2)^3 x^9 y^3 = -1760 x^9 y^3$$

Question 6:

Find the 13th term in the expansion of. $\left(9x - \dfrac{1}{3\sqrt{x}}\right)^{18}$, $x \neq 0$

Answer 6:

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r .$$

Thus, 13th term in the expansion of $\left(9x - \dfrac{1}{3\sqrt{x}}\right)^{18}$ is

$$T_{13} = T_{12+1} = {}^{18}C_{12}\left(9x\right)^{18-12}\left(-\dfrac{1}{3\sqrt{x}}\right)^{12}$$

$$= (-1)^{12}\dfrac{18!}{12!6!}(9)^{6}(x)^{6}\left(\dfrac{1}{3}\right)^{12}\left(\dfrac{1}{\sqrt{x}}\right)^{12}$$

$$= \dfrac{18\cdot17\cdot16\cdot15\cdot14\cdot13.12!}{12!.6\cdot5\cdot4\cdot3\cdot2}\cdot x^{6}\cdot\left(\dfrac{1}{x^{6}}\right).3^{12}\left(\dfrac{1}{3^{12}}\right) \qquad \left[9^{6} = \left(3^{2}\right)^{6} = 3^{12}\right]$$

$$= 18564$$

Question 7:
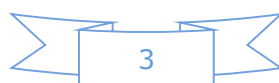Find the middle terms in the expansions of $\left(3 - \dfrac{x^{3}}{6}\right)^{7}$

Answer 7:

It is known that in the expansion of $(a + b)^{n}$, if *n* is odd, then there are two middle terms,

namely $\left(\dfrac{n+1}{2}\right)^{th}$ term and $\left(\dfrac{n+1}{2}+1\right)^{th}$ term.

Therefore, the middle terms in the expansion $\left(3 - \dfrac{x^{3}}{6}\right)^{7}$ are $\left(\dfrac{7+1}{2}\right)^{th} = 4^{th}$ term and $\left(\dfrac{7+1}{2}+1\right)^{th} = 5^{th}$ term

$$T_{4} = T_{3+1} = {}^{7}C_{3}(3)^{7-3}\left(-\dfrac{x^{3}}{6}\right)^{3} = (-1)^{3}\dfrac{7!}{3!4!}.3^{4}\cdot\dfrac{x^{9}}{6^{3}}$$

$$= -\dfrac{7\cdot6\cdot5.4!}{3\cdot2.4!}.3^{4}\cdot\dfrac{1}{2^{3}\cdot3^{3}}\cdot x^{9} = -\dfrac{105}{8}x^{9}$$

$$T_{5} = T_{4+1} = {}^{7}C_{4}(3)^{7-4}\left(-\dfrac{x^{3}}{6}\right)^{4} = (-1)^{4}\dfrac{7!}{4!3!}(3)^{3}\cdot\dfrac{x^{12}}{6^{4}}$$

$$= \dfrac{7\cdot6\cdot5.4!}{4!.3\cdot2}.\dfrac{3^{3}}{2^{4}\cdot3^{4}}\cdot x^{12} = \dfrac{35}{48}x^{12}$$

Thus, the middle terms in the expansion of $\left(3-\dfrac{x^3}{6}\right)^7$ are $-\dfrac{105}{8}x^9$ and $\dfrac{35}{48}x^{12}$

Question 8:

Find the middle terms in the expansions of $\left(\dfrac{x}{3}+9y\right)^{10}$

Answer 8:

It is known that in the expansion $(a + b)^n$, if n is even, then the middle term is $\left(\dfrac{n}{2}+1\right)^{th}$ term.

Therefore, the middle term in the expansion of $\left(\dfrac{x}{3}+9y\right)^{10}$ is $\left(\dfrac{10}{2}+1\right)^{th}=6^{th}$

$$T_6 = T_{5+1} = {}^{10}C_5\left(\dfrac{x}{3}\right)^{10-5}(9y)^5 = \dfrac{10!}{5!5!}\cdot\dfrac{x^5}{3^5}\cdot 9^5\cdot y^5$$

$$= \dfrac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5!}{5\cdot 4\cdot 3\cdot 2\cdot 5!}\cdot\dfrac{1}{3^5}\cdot 3^{10}\cdot x^5 y^5 \qquad \left[9^5 = \left(3^2\right)^5 = 3^{10}\right]$$

$$= 252\times 3^5\cdot x^5\cdot y^5 = 61236 x^5 y^5$$

Thus, the middle term in the expansion of $\left(\dfrac{x}{3}+9y\right)^{10}$ is $61236\,x^5 y^5$.

Question 9:

In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

Answer 9:

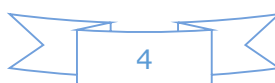It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that a^m occurs in the $(r + 1)^{th}$ term of the expansion $(1 + a)^{m+n}$,
we obtain

$$T_{r+1} = {}^{m+n}C_r\left(1\right)^{m+n-r}\left(a\right)^r = {}^{m+n}C_r a^r$$

Comparing the indices of a in a^m and in T_{r+1},

we obtain

$r = m$

Therefore, the coefficient of a^m is

$$^{m+n}C_m = \frac{(m+n)!}{m!(m+n-m)!} = \frac{(m+n)!}{m!n!} \qquad ...(1)$$

Assuming that a^n occurs in the $(k + 1)^{th}$ term of the expansion $(1 + a)^{m+n}$,

we obtain

$$T_{k+1} = {^{m+n}C_k}(1)^{m+n-k}(a)^k = {^{m+n}C_k}(a)^k$$

Comparing the indices of a in a^n and in T_{k+1},

we obtain

$k = n$

Therefore, the coefficient of a^n is

$$^{m+n}C_n = \frac{(m+n)!}{n!(m+n-n)!} = \frac{(m+n)!}{n!m!} \qquad ...(2)$$

Thus, from (1) and (2), it can be observed that the coefficients of a^m and a^n in the expansion of $(1 + a)^{m + n}$ are equal.

Question 10:

The coefficients of the $(r - 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are in the ratio 1:3:5. Find n and r.

Answer 10:

It is known that $(k + 1)^{th}$ term, (T_{k+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{k+1} = {^nC_k}a^{n-k}b^k$$

.

Therefore, $(r - 1)^{th}$ term in the expansion of $(x + 1)^n$ is

$$T_{r-1} = {^nC_{r-2}}(x)^{n-(r-2)}(1)^{(r-2)} = {^nC_{r-2}}x^{n-r+2}$$

$(r + 1)$ term in the expansion of $(x + 1)^n$ is

$$T_{r+1} = {^nC_r}(x)^{n-r}(1)^r = {^nC_r}x^{n-r}$$

Mathematics

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(Chapter – 8) (Binomial Theorem)

(Class – XI)

r^{th} term in the expansion of $(x + 1)^n$ is

$$T_r = {}^n C_{r-1}(x)^{n-(r-1)}(1)^{(r-1)} = {}^n C_{r-1}x^{n-r+1}$$

Therefore, the coefficients of the $(r - 1)^{th}$, r^{th}, and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ $\quad {}^n C_{r-2}, {}^n C_{r-1}$, and ${}^n C_r \quad$ are respectively. Since these coefficients are in the ratio 1:3:5, we obtain

$$\frac{{}^n C_{r-2}}{{}^n C_{r-1}} = \frac{1}{3} \text{ and } \frac{{}^n C_{r-1}}{{}^n C_r} = \frac{3}{5}$$

$$\frac{{}^n C_{r-2}}{{}^n C_{r-1}} = \frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{(r-1)(r-2)!(n-r+1)!}{(r-2)!(n-r+2)(n-r+1)!}$$

$$= \frac{r-1}{n-r+2}$$

$$\therefore \frac{r-1}{n-r+2} = \frac{1}{3}$$

$$\Rightarrow 3r - 3 = n - r + 2$$

$$\Rightarrow n - 4r + 5 = 0 \qquad \qquad \text{...(1)}$$

$$\frac{{}^n C_{r-1}}{{}^n C_r} = \frac{n!}{(r-1)!(n-r+1)} \times \frac{r!(n-r)!}{n!} = \frac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{r}{n-r+1}$$

$$\therefore \frac{r}{n-r+1} = \frac{3}{5}$$

$$\Rightarrow 5r = 3n - 3r + 3$$

$$\Rightarrow 3n - 8r + 3 = 0 \qquad \qquad \text{...(2)}$$

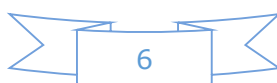Multiplying (1) by 3 and subtracting it from (2), we obtain

4*r* – 12 = 0

⇒ *r* = 3

Putting the value of *r* in (1), we obtain *n*

– 12 + 5 = 0

⇒ *n* = 7

Thus, *n* = 7 and *r* = 3

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Question 11:

Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Answer 11:

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that x^n occurs in the $(r + 1)^{th}$ term of the expansion of $(1 + x)^{2n}$, we obtain

$$T_{r+1} = {}^{2n}C_r (1)^{2n-r} (x)^r = {}^{2n}C_r (x)^r$$

Comparing the indices of x in x^n and in T_{r+1}, we obtain $r = n$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is

$${}^{2n}C_n = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n!)^2} \qquad \ldots(1)$$

Assuming that x^n occurs in the $(k+1)^{th}$ term of the expansion $(1 + x)^{2n-1}$, we obtain

$$T_{k+1} = {}^{2n-1}C_k (1)^{2n-1-k} (x)^k = {}^{2n-1}C_k (x)^k$$

Comparing the indices of x in x^n and T_{k+1}, we obtain $k = n$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$ is

$$\begin{aligned}
{}^{2n-1}C_n &= \frac{(2n-1)!}{n!(2n-1-n)!} = \frac{(2n-1)!}{n!(n-1)!} \\
&= \frac{2n.(2n-1)!}{2n.n!(n-1)!} = \frac{(2n)!}{2.n!n!} = \frac{1}{2}\left[\frac{(2n)!}{(n!)^2}\right] \qquad \ldots(2)
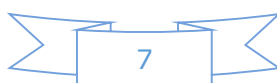\end{aligned}$$

From (1) and (2), it is observed that

$$\frac{1}{2}\left({}^{2n}C_n\right) = {}^{2n-1}C_n$$

$$\Rightarrow {}^{2n}C_n = 2\left({}^{2n-1}C_n\right)$$

Therefore,

the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Hence, proved.

Question 12:

Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Answer 12:

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that x^2 occurs in the $(r + 1)^{th}$ term of the expansion $(1 + x)^m$, we obtain

$$T_{r+1} = {}^mC_r (1)^{m-r} (x)^r = {}^mC_r (x)^r$$

Comparing the indices of x in x^2 and in T_{r+1}, we obtain $r = 2$

Therefore, the coefficient of x^2 is $\quad {}^mC_2$

It is given that the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

$$\therefore {}^mC_2 = 6$$
$$\Rightarrow \frac{m!}{2!(m-2)!} = 6$$
$$\Rightarrow \frac{m(m-1)(m-2)!}{2 \times (m-2)!} = 6$$
$$\Rightarrow m(m-1) = 12$$
$$\Rightarrow m^2 - m - 12 = 0$$
$$\Rightarrow m^2 - 4m + 3m - 12 = 0$$
$$\Rightarrow m(m-4) + 3(m-4) = 0$$
$$\Rightarrow (m-4)(m+3) = 0$$
$$\Rightarrow (m-4) = 0 \text{ or } (m+3) = 0$$
$$\Rightarrow m = 4 \text{ or } m = -3$$

Thus, the positive value of m, for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6, is 4.