

Mathematics

(www.tiwariacademy.com : A step towards free education)

(Chapter – 8) (Binomial Theorem)

(Class – XI)

Miscellaneous Exercise on Chapter 8

Question 1:

Find a , b and n in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Answer 1:

It is known that $(r + 1)^{\text{th}}$ term, (T_{r+1}) , in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^n C_r a^{n-r} b^r$$

The first three terms of the expansion are given as 729, 7290, and 30375 respectively.

Therefore, we obtain

$$T_1 = {}^n C_0 a^{n-0} b^0 = a^n = 729 \quad \dots(1)$$

$$T_2 = {}^n C_1 a^{n-1} b^1 = na^{n-1}b = 7290 \quad \dots(2)$$

$$T_3 = {}^n C_2 a^{n-2} b^2 = \frac{n(n-1)}{2} a^{n-2} b^2 = 30375 \quad \dots(3)$$

Dividing (2) by (1), we obtain

$$\begin{aligned} \frac{na^{n-1}b}{a^n} &= \frac{7290}{729} \\ \Rightarrow \frac{nb}{a} &= 10 \quad \dots(4) \end{aligned}$$

Dividing (3) by (2), we obtain

$$\begin{aligned} \frac{n(n-1)a^{n-2}b^2}{2na^{n-1}b} &= \frac{30375}{7290} \\ \Rightarrow \frac{(n-1)b}{2a} &= \frac{30375}{7290} \\ \Rightarrow \frac{(n-1)b}{a} &= \frac{30375 \times 2}{7290} = \frac{25}{3} \\ \Rightarrow \frac{nb}{a} - \frac{b}{a} &= \frac{25}{3} \\ \Rightarrow 10 - \frac{b}{a} &= \frac{25}{3} \quad [\text{Using (4)}] \\ \Rightarrow \frac{b}{a} &= 10 - \frac{25}{3} = \frac{5}{3} \quad \dots(5) \end{aligned}$$



Mathematics

(www.tiwariacademy.com : A step towards free education)

(Chapter – 8) (Binomial Theorem)

(Class – XI)

From (4) and (5), we obtain

$$n \cdot \frac{5}{3} = 10$$

$$\Rightarrow n = 6$$

Substituting $n = 6$ in equation (1), we obtain a^6

$$= 729$$

$$\Rightarrow a = \sqrt[6]{729} = 3$$

From (5), we obtain

$$\frac{b}{3} = \frac{5}{3} \Rightarrow b = 5$$

Thus, $a = 3$, $b = 5$, and $n = 6$.

Question 2:

Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.

Answer 2:

It is known that $(r + 1)^{\text{th}}$ term, (T_{r+1}) , in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^n C_r a^{n-r} b^r$$

Assuming that x^2 occurs in the $(r + 1)^{\text{th}}$ term in the expansion of $(3 + ax)^9$, we obtain

$$T_{r+1} = {}^9 C_r (3)^{9-r} (ax)^r = {}^9 C_r (3)^{9-r} a^r x^r$$

Comparing the indices of x in x^2 and in T_{r+1} , we obtain

$$r = 2$$

Thus, the coefficient of x^2 is

$${}^9 C_2 (3)^{9-2} a^2 = \frac{9!}{2!7!} (3)^7 a^2 = 36(3)^7 a^2$$

Assuming that x^3 occurs in the $(k + 1)^{\text{th}}$ term in the expansion of $(3 + ax)^9$, we obtain

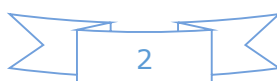
$$T_{k+1} = {}^9 C_k (3)^{9-k} (ax)^k = {}^9 C_k (3)^{9-k} a^k x^k$$

Comparing the indices of x in x^3 and in T_{k+1} , we obtain $k = 3$

Thus, the coefficient of x^3 is

$${}^9 C_3 (3)^{9-3} a^3 = \frac{9!}{3!6!} (3)^6 a^3 = 84(3)^6 a^3$$

It is given that the coefficients of x^2 and x^3 are the same.



Mathematics

(www.tiwariacademy.com : A step towards free education)

(Chapter – 8) (Binomial Theorem)

(Class – XI)

$$84(3)^6 a^3 = 36(3)^7 a^2$$

$$\Rightarrow 84a = 36 \times 3$$

$$\Rightarrow a = \frac{36 \times 3}{84} = \frac{104}{84}$$

$$\Rightarrow a = \frac{9}{7}$$

Thus, the required value of a is $9/7$.

Question 3:

Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 - x)^7$ using binomial theorem.

Answer 3:

Using Binomial Theorem, the expressions, $(1 + 2x)^6$ and $(1 - x)^7$, can be expanded as

$$(1+2x)^6 = {}^6C_0 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 \\ + {}^6C_5(2x)^5 + {}^6C_6(2x)^6$$

$$= 1 + 6(2x) + 15(2x)^2 + 20(2x)^3 + 15(2x)^4 + 6(2x)^5 + (2x)^6$$

$$= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$$

$$(1-x)^7 = {}^7C_0 - {}^7C_1(x) + {}^7C_2(x)^2 - {}^7C_3(x)^3 + {}^7C_4(x)^4$$

$$- {}^7C_5(x)^5 + {}^7C_6(x)^6 - {}^7C_7(x)^7$$

$$= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$$

$$\therefore (1+2x)^6 (1-x)^7$$

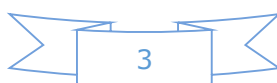
$$= (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6)(1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7)$$

The complete multiplication of the two brackets is not required to be carried out. Only those terms, which involve x^5 , are required.

The terms containing x^5 are

$$1(-21x^5) + (12x)(35x^4) + (60x^2)(-35x^3) + (160x^3)(21x^2) + (240x^4)(-7x) + (192x^5)(1) \\ = 171x^5$$

Thus, the coefficient of x^5 in the given product is 171.



Mathematics

(www.tiwariacademy.com : A step towards free education)

(Chapter – 8) (Binomial Theorem)

(Class – XI)

Question 4:

If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer. [Hint: write $a^n = (a - b + b)^n$ and expand]

Answer 4:

In order to prove that $(a - b)$ is a factor of $(a^n - b^n)$, it has to be proved that

$a^n - b^n = k(a - b)$, where k is some natural number

It can be written that, $a = a - b + b$

$$\begin{aligned}\therefore a^n &= (a - b + b)^n = [(a - b) + b]^n \\ &= {}^n C_0 (a - b)^n + {}^n C_1 (a - b)^{n-1} b + \dots + {}^n C_{n-1} (a - b) b^{n-1} + {}^n C_n b^n \\ &= (a - b)^n + {}^n C_1 (a - b)^{n-1} b + \dots + {}^n C_{n-1} (a - b) b^{n-1} + b^n \\ \Rightarrow a^n - b^n &= (a - b) [(a - b)^{n-1} + {}^n C_1 (a - b)^{n-2} b + \dots + {}^n C_{n-1} b^{n-1}] \\ \Rightarrow a^n - b^n &= k(a - b)\end{aligned}$$

where, $k = [(a - b)^{n-1} + {}^n C_1 (a - b)^{n-2} b + \dots + {}^n C_{n-1} b^{n-1}]$ is a natural number

This shows that $(a - b)$ is a factor of $(a^n - b^n)$, where n is a positive integer.

Question 5:

Evaluate. $(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6$

Answer 5:

Firstly, the expression $(a + b)^6 - (a - b)^6$ is simplified by using Binomial Theorem. This can be done as

$$\begin{aligned}(a + b)^6 &= {}^6 C_0 a^6 + {}^6 C_1 a^5 b + {}^6 C_2 a^4 b^2 + {}^6 C_3 a^3 b^3 + {}^6 C_4 a^2 b^4 + {}^6 C_5 a b^5 + {}^6 C_6 b^6 \\ &= a^6 + 6a^5 b + 15a^4 b^2 + 20a^3 b^3 + 15a^2 b^4 + 6ab^5 + b^6 \\ (a - b)^6 &= {}^6 C_0 a^6 - {}^6 C_1 a^5 b + {}^6 C_2 a^4 b^2 - {}^6 C_3 a^3 b^3 + {}^6 C_4 a^2 b^4 - {}^6 C_5 a b^5 + {}^6 C_6 b^6 \\ &= a^6 - 6a^5 b + 15a^4 b^2 - 20a^3 b^3 + 15a^2 b^4 - 6ab^5 + b^6 \\ \therefore (a + b)^6 - (a - b)^6 &= 2[6a^5 b + 20a^3 b^3 + 6ab^5]\end{aligned}$$

Putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$\begin{aligned}(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6 &= 2[6(\sqrt{3})^5 (\sqrt{2}) + 20(\sqrt{3})^3 (\sqrt{2})^3 + 6(\sqrt{3})(\sqrt{2})^5] \\ &= 2[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}] \\ &= 2 \times 198\sqrt{6} \\ &= 396\sqrt{6}\end{aligned}$$

Mathematics

(www.tiwariacademy.com : A step towards free education)

(Chapter – 8) (Binomial Theorem)

(Class – XI)

Question 6:

Find the value of $(a^2 + \sqrt{a^2 - 1})^4 + (a^2 - \sqrt{a^2 - 1})^4$

Answer 6:

Firstly, the expression $(x + y)^4 + (x - y)^4$ is simplified by using Binomial Theorem.

This can be done as

$$\begin{aligned}(x + y)^4 &= {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

$$\begin{aligned}(x - y)^4 &= {}^4C_0x^4 - {}^4C_1x^3y + {}^4C_2x^2y^2 - {}^4C_3xy^3 + {}^4C_4y^4 \\ &= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4\end{aligned}$$

$$\therefore (x + y)^4 + (x - y)^4 = 2(x^4 + 6x^2y^2 + y^4)$$

Putting $x = a^2$ and $y = \sqrt{a^2 - 1}$, we obtain

$$\begin{aligned}(a^2 + \sqrt{a^2 - 1})^4 + (a^2 - \sqrt{a^2 - 1})^4 &= 2 \left[(a^2)^4 + 6(a^2)^2 (\sqrt{a^2 - 1})^2 + (\sqrt{a^2 - 1})^4 \right] \\ &= 2 \left[a^8 + 6a^4 (a^2 - 1) + (a^2 - 1)^2 \right] \\ &= 2 \left[a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1 \right] \\ &= 2 \left[a^8 + 6a^6 - 5a^4 - 2a^2 + 1 \right] \\ &= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2\end{aligned}$$

Question 7:

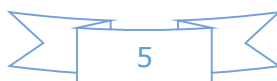
Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Answer 7:

$$0.99 = 1 - 0.01$$

$$\begin{aligned}\therefore (0.99)^5 &= (1 - 0.01)^5 \\ &= {}^5C_0(1)^5 - {}^5C_1(1)^4(0.01) + {}^5C_2(1)^3(0.01)^2 \quad \text{(Approximately)} \\ &= 1 - 5(0.01) + 10(0.01)^2 \\ &= 1 - 0.05 + 0.001 \\ &= 1.001 - 0.05 \\ &= 0.951\end{aligned}$$

Thus, the value of $(0.99)^5$ is approximately 0.951.



Mathematics

(www.tiwariacademy.com : A step towards free education)

(Chapter – 8) (Binomial Theorem)

(Class – XI)

Question 8:

Find n , if the ratio of the fifth term from the beginning to the fifth term from the end in

the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6} : 1$

Answer 8:

In the expansion, $(a + b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_{n-1} a b^{n-1} + {}^nC_n b^n$

Fifth term from the beginning = ${}^nC_4 a^{n-4} b^4$

Fifth term from the end = ${}^nC_{n-4} a^4 b^{n-4}$

Therefore, it is evident that in the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ the fifth term from the

beginning is ${}^nC_4 (\sqrt[4]{2})^{n-4} \left(\frac{1}{\sqrt[4]{3}}\right)^4$ and the fifth term from the end is ${}^nC_{n-4} (\sqrt[4]{2})^4 \left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$

$${}^nC_4 (\sqrt[4]{2})^{n-4} \left(\frac{1}{\sqrt[4]{3}}\right)^4 = {}^nC_4 \frac{(\sqrt[4]{2})^n}{(\sqrt[4]{2})^4} \cdot \frac{1}{3} = {}^nC_4 \frac{(\sqrt[4]{2})^n}{2} \cdot \frac{1}{3} = \frac{n!}{6 \cdot 4! (n-4)!} (\sqrt[4]{2})^n \quad \dots(1)$$

$${}^nC_{n-4} (\sqrt[4]{2})^4 \left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = {}^nC_{n-4} \cdot 2 \cdot \frac{(\sqrt[4]{3})^4}{(\sqrt[4]{3})^n} = {}^nC_{n-4} \cdot 2 \cdot \frac{3}{(\sqrt[4]{3})^n} = \frac{6n!}{(n-4)! 4!} \cdot \frac{1}{(\sqrt[4]{3})^n} \quad \dots(2)$$

It is given that the ratio of the fifth term from the beginning to the fifth term from the end is $\sqrt{6} : 1$. Therefore, from (1) and (2), we obtain

$$\frac{n!}{6 \cdot 4! (n-4)!} (\sqrt[4]{2})^n : \frac{6n!}{(n-4)! 4!} \cdot \frac{1}{(\sqrt[4]{3})^n} = \sqrt{6} : 1$$

$$\Rightarrow \frac{(\sqrt[4]{2})^n}{6} : \frac{6}{(\sqrt[4]{3})^n} = \sqrt{6} : 1$$

$$\Rightarrow \frac{(\sqrt[4]{2})^n}{6} \times \frac{(\sqrt[4]{3})^n}{6} = \sqrt{6}$$

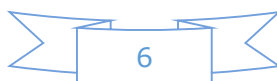
$$\Rightarrow (\sqrt[4]{6})^n = 36\sqrt{6}$$

$$\Rightarrow 6^{\frac{n}{4}} = 6^{\frac{5}{2}}$$

$$\Rightarrow \frac{n}{4} = \frac{5}{2}$$

$$\Rightarrow n = 4 \times \frac{5}{2} = 10$$

Thus, the value of n is 10.



Mathematics

(www.tiwariacademy.com : A step towards free education)

(Chapter – 8) (Binomial Theorem)

(Class – XI)

Question 9:

Expand using Binomial Theorem $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4, x \neq 0$

Answer 9:

Using Binomial Theorem, the given expression $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4$ can be expanded as

$$\begin{aligned} & \left[\left(1 + \frac{x}{2}\right) - \frac{2}{x}\right]^4 \\ &= {}^4C_0 \left(1 + \frac{x}{2}\right)^4 - {}^4C_1 \left(1 + \frac{x}{2}\right)^3 \left(\frac{2}{x}\right) + {}^4C_2 \left(1 + \frac{x}{2}\right)^2 \left(\frac{2}{x}\right)^2 - {}^4C_3 \left(1 + \frac{x}{2}\right) \left(\frac{2}{x}\right)^3 + {}^4C_4 \left(\frac{2}{x}\right)^4 \\ &= \left(1 + \frac{x}{2}\right)^4 - 4 \left(1 + \frac{x}{2}\right)^3 \left(\frac{2}{x}\right) + 6 \left(1 + x + \frac{x^2}{4}\right) \left(\frac{4}{x^2}\right) - 4 \left(1 + \frac{x}{2}\right) \left(\frac{8}{x^3}\right) + \frac{16}{x^4} \\ &= \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x} \left(1 + \frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4} \\ &= \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x} \left(1 + \frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \end{aligned} \quad \dots(1)$$

Again by using Binomial Theorem, we obtain

$$\begin{aligned} \left(1 + \frac{x}{2}\right)^4 &= {}^4C_0 (1)^4 + {}^4C_1 (1)^3 \left(\frac{x}{2}\right) + {}^4C_2 (1)^2 \left(\frac{x}{2}\right)^2 + {}^4C_3 (1) \left(\frac{x}{2}\right)^3 + {}^4C_4 \left(\frac{x}{2}\right)^4 \\ &= 1 + 4 \times \frac{x}{2} + 6 \times \frac{x^2}{4} + 4 \times \frac{x^3}{8} + \frac{x^4}{16} \\ &= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} \end{aligned} \quad \dots(2)$$

$$\begin{aligned} \left(1 + \frac{x}{2}\right)^3 &= {}^3C_0 (1)^3 + {}^3C_1 (1)^2 \left(\frac{x}{2}\right) + {}^3C_2 (1) \left(\frac{x}{2}\right)^2 + {}^3C_3 \left(\frac{x}{2}\right)^3 \\ &= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \end{aligned} \quad \dots(3)$$

From (1), (2), and (3), we obtain



Mathematics

(www.tiwariacademy.com : A step towards free education)

(Chapter – 8) (Binomial Theorem)

(Class – XI)

$$\begin{aligned} & \left[\left(1 + \frac{x}{2} \right) - \frac{2}{x} \right]^4 \\ &= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} \left(1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \right) + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \\ &= 1 + 2x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \\ &= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5 \end{aligned}$$

Question 10:

Find the expansion of $(3x^2 - 2ax + 3a^2)^3$ using binomial theorem.

Answer 10:

Using Binomial Theorem, the given expression $(3x^2 - 2ax + 3a^2)^3$ can be expanded as

$$\begin{aligned} & \left[(3x^2 - 2ax) + 3a^2 \right]^3 \\ &= {}^3C_0 (3x^2 - 2ax)^3 + {}^3C_1 (3x^2 - 2ax)^2 (3a^2) + {}^3C_2 (3x^2 - 2ax)(3a^2)^2 + {}^3C_3 (3a^2)^3 \\ &= (3x^2 - 2ax)^3 + 3(9x^4 - 12ax^3 + 4a^2x^2)(3a^2) + 3(3x^2 - 2ax)(9a^4) + 27a^6 \\ &= (3x^2 - 2ax)^3 + 81a^2x^4 - 108a^3x^3 + 36a^4x^2 + 81a^4x^2 - 54a^5x + 27a^6 \\ &= (3x^2 - 2ax)^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6 \quad \dots(1) \end{aligned}$$

Again by using Binomial Theorem, we obtain

$$\begin{aligned} & (3x^2 - 2ax)^3 \\ &= {}^3C_0 (3x^2)^3 - {}^3C_1 (3x^2)^2 (2ax) + {}^3C_2 (3x^2)(2ax)^2 - {}^3C_3 (2ax)^3 \\ &= 27x^6 - 3(9x^4)(2ax) + 3(3x^2)(4a^2x^2) - 8a^3x^3 \\ &= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 \quad \dots(2) \end{aligned}$$

From (1) and (2), we obtain

$$\begin{aligned} & (3x^2 - 2ax + 3a^2)^3 \\ &= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6 \\ &= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6 \end{aligned}$$

