Exercise 9.4

Question 1:

Find the sum to n terms of the series $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \ldots$

Answer 1:

The given series is $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times$

$5 + \ldots$ n^{th} term, $a_n = n\,(n + 1)$

$$\therefore S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k(k+1)$$
$$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\left(\frac{2n+1}{3}+1\right)$$
$$= \frac{n(n+1)}{2}\left(\frac{2n+4}{3}\right)$$
$$= \frac{n(n+1)(n+2)}{3}$$

Question 2:

Find the sum to n terms of the series $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \ldots$

Answer 2:

The given series is $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \ldots$ n^{th} term,

$a_n = n\,(n + 1)\,(n + 2)$

$= (n^2 + n)\,(n + 2)$

$= n^3 + 3n^2 + 2n$

$$\therefore S_n = \sum_{k=1}^{n} a_k$$

$$= \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{2} + n(n+1)$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + 2n + 1 + 2\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{n^2 + n + 4n + 6}{2}\right]$$

$$= \frac{n(n+1)}{4}\left(n^2 + 5n + 6\right)$$

$$= \frac{n(n+1)}{4}\left(n^2 + 2n + 3n + 6\right)$$

$$= \frac{n(n+1)\left[n(n+2) + 3(n+2)\right]}{4}$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

Question 3:

Find the sum to n terms of the series $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots$

Answer 3:

The given series is $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots$ n^{th} term,

$a_n = (2n + 1)\, n^2 = 2n^3 + n^2$

$$\therefore S_n = \sum_{k=1}^{n} a_k$$

$$= \sum_{k=1}^{n} = \left(2k^3 + k^2\right) = 2\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2$$

$$= 2\left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n^2(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left[n(n+1) + \frac{2n+1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2 + 3n + 2n + 1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2 + 5n + 1}{3}\right]$$

$$= \frac{n(n+1)\left(3n^2 + 5n + 1\right)}{6}$$

Question 4:

Find the sum to n terms of the series $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots$

Answer 4:

The given series is $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots$

n^{th} term, $a_n = \frac{1}{n(n+1)} \quad = \frac{1}{n} - \frac{1}{n+1} \qquad$ (By partial fractions)

$$a_1 = \frac{1}{1} - \frac{1}{2}$$

$$a_2 = \frac{1}{2} - \frac{1}{3}$$

$$a_3 = \frac{1}{3} - \frac{1}{4} \ldots$$

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

Adding the above terms column wise, we obtain

$$a_1 + a_2 + \ldots + a_n = \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}\right] - \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \frac{1}{n+1}\right]$$

$$\therefore S_n = 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

Question 5:

Find the sum to n terms of the series $5^2 + 6^2 + 7^2 + \ldots + 20^2$

Answer 5:

The given series is $5^2 + 6^2 + 7^2 + \ldots + 20^2$ n^{th} term,

$a_n = (n + 4)^2 = n^2 + 8n + 16$

$$\therefore S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(k^2 + 8k + 16\right)$$

$$= \sum_{k=1}^{n} k^2 + 8\sum_{k=1}^{n} k + \sum_{k=1}^{n} 16$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{8n(n+1)}{2} + 16n$$

16^{th} term is $(16 + 4)^2 = 20^2$

$$\therefore S_{16} = \frac{16(16+1)(2\times16+1)}{6} + \frac{8\times16\times(16+1)}{2} + 16\times16$$
$$= \frac{(16)(17)(33)}{6} + \frac{(8)\times16\times(16+1)}{2} + 16\times16$$
$$= \frac{(16)(17)(33)}{6} + \frac{(8)(16)(17)}{2} + 256$$
$$= 1496 + 1088 + 256$$
$$= 2840$$
$$\therefore 5^2 + 6^2 + 7^2 + \ldots\ldots + 20^2 = 2840$$

Question 6:

Find the sum to n terms of the series 3 × 8 + 6 × 11 + 9 × 14 +…

Answer 6:

The given series is 3 × 8 + 6 × 11 + 9 × 14 + … a_n

= (n^{th} term of 3, 6, 9 …) × (n^{th} term of 8, 11, 14 …)

= $(3n)(3n + 5)$

= $9n^2 + 15n$

$$\therefore S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(9k^2 + 15k\right)$$
$$= 9\sum_{k=1}^{n} k^2 + 15\sum_{k=1}^{n} k$$
$$= 9\times\frac{n(n+1)(2n+1)}{6} + 15\times\frac{n(n+1)}{2}$$
$$= \frac{3n(n+1)(2n+1)}{2} + \frac{15n(n+1)}{2}$$
$$= \frac{3n(n+1)}{2}(2n+1+5)$$
$$= \frac{3n(n+1)}{2}(2n+6)$$
$$= 3n(n+1)(n+3)$$

Question 7:

Find the sum to *n* terms of the series $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$

Answer 7:

The given series is $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^3\ ) + \ldots\ a_n$

$= (1^2 + 2^2 + 3^3 + \ldots\ldots.+ n^2)$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(2n^2+3n+1)}{6} = \frac{2^3+3n^2+n}{6}$$

$$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$\therefore S_n = \sum_{k=1}^{n} a_k$$

$$= \sum_{k=1}^{n}\left(\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k\right)$$

$$= \frac{1}{3}\sum_{k=1}^{n} k^3 + \frac{1}{2}\sum_{k=1}^{n} k^2 + \frac{1}{6}\sum_{k=1}^{n} k$$

$$= \frac{1}{3}\frac{n^2(n+1)^2}{(2)^2} + \frac{1}{2}\times\frac{n(n+1)(2n+1)}{6} + \frac{1}{6}\times\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{6}\left[\frac{n(n+1)}{2} + \frac{(2n+1)}{2} + \frac{1}{2}\right]$$

$$= \frac{n(n+1)}{6}\left[\frac{n^2+n+2n+1+1}{2}\right]$$

$$= \frac{n(n+1)}{6}\left[\frac{n^2+n+2n+2}{2}\right]$$

$$= \frac{n(n+1)}{6}\left[\frac{n(n+1)+2(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{6}\left[\frac{(n+1)(n+2)}{2}\right]$$

$$= \frac{n(n+1)^2(n+2)}{12}$$

Question 8:

Find the sum to n terms of the series whose n^{th} term is given by $n(n+1)(n+4)$.

Answer 8:

$a_n = n(n+1)(n+4) = n(n^2+5n+4) = n^3+5n^2+4n$

$$\therefore S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^3 + 5\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k$$

$$= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{5(2n+1)}{3} + 4\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+20n+10+24}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+23n+34}{6}\right]$$

$$= \frac{n(n+1)(3n^2+23n+34)}{12}$$

Question 9:

Find the sum to n terms of the series whose n^{th} terms is given by $n^2 + 2^n$

Answer 9:

$a_n = n^2 + 2^n$

$$\therefore S_n = \sum_{k=1}^{n} k^2 + 2^k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k \qquad (1)$$

Consider $\sum_{k=1}^{n} 2^k = 2^1 + 2^2 + 2^3 + \ldots$

The above series 2, 2^2, 2^3 … is a G.P. with both the first term and common ratio equal to 2.

$$\therefore \sum_{k=1}^{n} 2^k = \frac{(2)\left[(2)^n - 1\right]}{2-1} = 2\left(2^n - 1\right) \qquad (2)$$

Therefore, from (1) and (2), we obtain

$$S_n = \sum_{k=1}^{n} k^2 + 2\left(2^n - 1\right) = \frac{n(n+1)(2n+1)}{6} + 2\left(2^n - 1\right)$$

Question 10:

Find the sum to n terms of the series whose n^{th} terms is given by $(2n - 1)^2$

Answer 10:

$a_n = (2n - 1)^2 = 4n^2 - 4n + 1$

$$\therefore S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(4k^2 - 4k + 1\right)$$

$$= 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$

$$= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n$$

$$= n\left[\frac{2(2n^2+3n+1)}{3} - 2(n+1) + 1\right]$$

$$= n\left[\frac{4n^2+6n+2-6n-6+3}{3}\right]$$

$$= n\left[\frac{4n^2-1}{3}\right]$$

$$= \frac{n(2n+1)(2n-1)}{3}$$