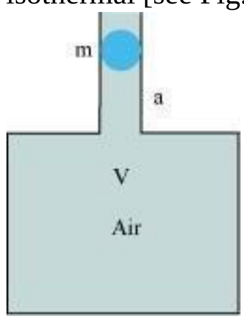
(Chapter 15)(Oscillations)
XI

Additional Exercises

Question 14.20:
An air chamber of volume V has a neck area of cross section a into which a ball of mass m just fits and can move up and down without any friction (Fig.14.33). Show that when the ball is pressed down a little and released, it executes SHM. Obtain an expression for the time period of oscillations assuming pressure-volume variations of air to be isothermal [see Fig. 14.33].



Answer
Volume of the air chamber = V

Area of cross-section of the neck = a

Mass of the ball = m

The pressure inside the chamber is equal to the atmospheric pressure.

Let the ball be depressed by x units. As a result of this depression, there would be a decrease in the volume and an increase in the pressure inside the chamber.

Decrease in the volume of the air chamber, $\Delta V = ax$

Volumetric strain $= \dfrac{\text{Change in volume}}{\text{Original volume}}$

$$\Rightarrow \frac{\Delta V}{V} = \frac{ax}{V}$$

Bulk Modulus of air, $B = \frac{\text{Stress}}{\text{Strain}} = \frac{-p}{\frac{ax}{V}}$

In this case, stress is the increase in pressure. The negative sign indicates that pressure increases with a decrease in volume.

$$p = \frac{-Bax}{V}$$

The restoring force acting on the ball,

$$F = p \times a$$

$$= \frac{-Bax}{V} \cdot a$$

$$= \frac{-Ba^2x}{V} \qquad \dots (i)$$

In simple harmonic motion, the equation for restoring force is:

$F = -kx$ … (ii)

Where, k is the spring constant

Comparing equations (i) and (ii), we get:

$$k = \frac{Ba^2}{V}$$

Time period,

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$= 2\pi\sqrt{\frac{Vm}{Ba^2}}$$

Question 14.21:
You are riding in an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of (a) the spring constant *k* and (b) the damping constant *b* for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg.

Answer
Mass of the automobile, m = 3000 kg

Displacement in the suspension system, x = 15 cm = 0.15 m

There are 4 springs in parallel to the support of the mass of the automobile.

The equation for the restoring force for the system:

$F = -4kx = mg$

Where, k is the spring constant of the suspension system

Time period, $T = 2\pi\sqrt{\frac{m}{4k}}$

And $k = \frac{mg}{4x} = \frac{3000 \times 10}{4 \times 0.15}$ = 5000 = 5×10^4 N/m

Spring constant, $k = 5 \times 10^4$ N/m

Each wheel supports a mass, $M = \frac{3000}{4} = 750$ kg

For damping factor b, the equation for displacement is written as:

$$x = x_0 e^{-bt/2M}$$

The amplitude of oscillation decreases by 50%.

$$\therefore x = \frac{x_0}{2}$$

$$\frac{x_0}{2} = x_0 e^{-bt/2M}$$

$$\log_e 2 = \frac{bt}{2M}$$

$$\therefore b = \frac{2M \log_e 2}{t}$$

Where,

Time period, $t = 2\pi\sqrt{\frac{m}{4k}} = 2\pi\sqrt{\frac{3000}{4 \times 5 \times 10^4}} = 0.7691$ s

$$\therefore b = \frac{2 \times 750 \times 0.693}{0.7691} = 1351.58 \text{ kg/s}$$

Therefore, the damping constant of the spring is 1351.58 kg/s.

Question 14.22:
Show that for a particle in linear SHM the average kinetic energy over a period of oscillation equals the average potential energy over the same period.

Answer
The equation of displacement of a particle executing SHM at an instant t is given as:
$x = A\sin\omega t$

Where,

A = Amplitude of oscillation

ω = Angular frequency $= \sqrt{\frac{k}{M}}$

The velocity of the particle is:

$$v = \frac{dx}{dt} = A\omega\cos\omega t$$

The kinetic energy of the particle is:
$$E_k = \frac{1}{2}Mv^2 = \frac{1}{2}MA^2\omega^2\cos^2\omega t$$

The potential energy of the particle is:
$$E_p = \frac{1}{2}kx^2 = \frac{1}{2}M\omega^2A^2\sin^2\omega t$$

For time period T, the average kinetic energy over a single cycle is given as:

$$(E_k)_{\text{Avg}} = \frac{1}{T}\int_0^T E_k \, dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2} MA^2\omega^2 \cos^2 \omega t \, dt$$

$$= \frac{1}{2T} MA^2\omega^2 \int_0^T \frac{(1+\cos 2\omega t)}{2} dt$$

$$= \frac{1}{4T} MA^2\omega^2 \left[t + \frac{\sin 2\omega t}{2\omega} \right]_0^T$$

$$= \frac{1}{4T} MA^2\omega^2 (T)$$

$$= \frac{1}{4} MA^2\omega^2 \qquad \ldots (i)$$

And, average potential energy over one cycle is given as:

$$(E_p)_{\text{Avg}} = \frac{1}{T}\int_0^T E_p \, dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2} M\omega^2 A^2 \sin^2 \omega t \, dt$$

$$= \frac{1}{2T} M\omega^2 A^2 \int_0^T \frac{(1-\cos 2\omega t)}{2} dt$$

$$= \frac{1}{4T} M\omega^2 A^2 \left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^T$$

$$= \frac{1}{4T} M\omega^2 A^2 (T)$$

$$= \frac{M\omega^2 A^2}{4} \quad \ldots (ii)$$

It can be inferred from equations (*i*) and (*ii*) that the average kinetic energy for a given time period is equal to the average potential energy for the same time period.

Question 14.23:
A circular disc of mass 10 kg is suspended by a wire attached to its centre. The wire is twisted by rotating the disc and released. The period of torsional oscillations is found to be 1.5 s. The radius of the disc is 15 cm. Determine the torsional spring constant of the wire. (Torsional spring constant α is defined by the relation $J = -\alpha\, \theta$, where J is the restoring couple and θ the angle of twist).

Answer
Mass of the circular disc, m = 10 kg

Radius of the disc, r = 15 cm = 0.15 m

The torsional oscillations of the disc has a time period, T = 1.5 s

The moment of inertia of the disc is:

$$I = \frac{1}{2} m r^2$$

$$= \frac{1}{2} \times (10) \times (0.15)^2$$

$= 0.1125$ kg m^2

Time period, $T = 2\pi\sqrt{\frac{I}{\alpha}}$ α

is the torsional constant.

$$\alpha = \frac{4\pi^2 I}{T^2}$$

$$= \frac{4 \times (\pi)^2 \times 0.1125}{(1.5)^2}$$

= 1.972 Nm/rad

Hence, the torsional spring constant of the wire is 1.972 Nm rad^{-1}.

Question 14.24:
A body describes simple harmonic motion with amplitude of 5 cm and a period of 0.2 s. Find the acceleration and velocity of the body when the displacement is (a) 5 cm, (b) 3 cm, (c) 0 cm.

Answer

Amplitude, A = 5 cm = 0.05 m

Time period, T = 0.2 s

For displacement, x = 5 cm = 0.05 m

Acceleration is given by:

$$a = -\omega^2 x$$

$$= -\left(\frac{2\pi}{T}\right)^2 x$$

$$= -\left(\frac{2\pi}{0.2}\right)^2 \times 0.05$$

$$= -5\pi^2 \text{ m/s}^2$$

Velocity is given by:

$$v = \omega\sqrt{A^2 - x^2}$$

$$= \frac{2\pi}{T}\sqrt{(0.05)^2 - (0.05)^2}$$

= 0

When the displacement of the body is 5 cm, its acceleration is $-5\pi^2$ m/s^2 and velocity is 0.

For displacement, x = 3 cm = 0.03 m

Acceleration is given by:

$$a = -\omega^2 x$$

$$= -\left(\frac{2\pi}{T}\right)^2 x$$

$$= -\left(\frac{2\pi}{0.2}\right)^2 0.03$$

$$= -3\pi^2 \text{ m/s}^2$$

Velocity is given by:

$$v = \omega\sqrt{A^2 - x^2}$$

$$= \frac{2\pi}{T}\sqrt{A^2 - x^2}$$

$$= \frac{2\pi}{T}\sqrt{(0.05)^2 - (0.03)^2}$$

$$= \frac{2\pi}{0.2} \times 0.04$$

$= 0.4\ \pi$ m/s

When the displacement of the body is 3 cm, its acceleration is -3π m/s^2 and velocity is 0.4π m/s.

For displacement, $x = 0$

Acceleration is given by:

$a = -\omega^2 x = 0$

Velocity is given by:

$v = \omega\sqrt{A^2 - x^2}$

$= \frac{2\pi}{T}\sqrt{A^2 - x^2}$

$= \frac{2\pi}{0.2}\sqrt{(0.05)^2 - 0}$

$= 0.5\pi$ m/s

When the displacement of the body is 0, its acceleration is 0 and velocity is 0.5 π m/s.

Question 14.25:
A mass attached to a spring is free to oscillate, with angular velocity ω, in a horizontal plane without friction or damping. It is pulled to a distance x_0 and pushed towards the centre with a velocity v_0 at time $t = 0$. Determine the amplitude of the resulting oscillations in terms of the parameters ω, x_0 and v_0. [Hint: Start with the equation $x = a \cos(\omega t+\theta)$ and note that the initial velocity is negative.]

Answer
The displacement equation for an oscillating mass is given by:

$x = A\cos(\omega t + \theta)$

Where,

A is the amplitude x

is the displacement θ

is the phase constant

Velocity, $v = \frac{dx}{dt} = -A\omega \sin(\omega t + \theta)$

At $t = 0$, $x = x_0$ $x_0 =$

$A\cos\theta = x_0$ … (i)

And, $\frac{dx}{dt} = -v_0 = A\omega \sin\theta$

$A\sin\theta = \frac{v_0}{\omega}$ … (ii)

Squaring and adding equations (i) and (ii), we get:

$$A^2(\cos^2\theta + \sin^2\theta) = x_0^2 + \left(\frac{v_0^2}{\omega^2}\right)$$

$$\therefore A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$$

Hence, the amplitude of the resulting oscillation is $\sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$.