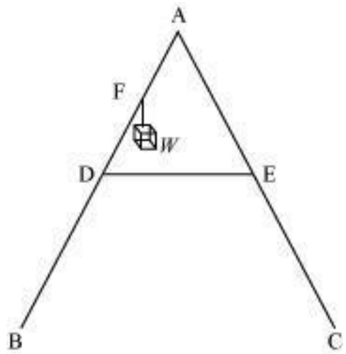
Additional Exercises

Question 7.22:

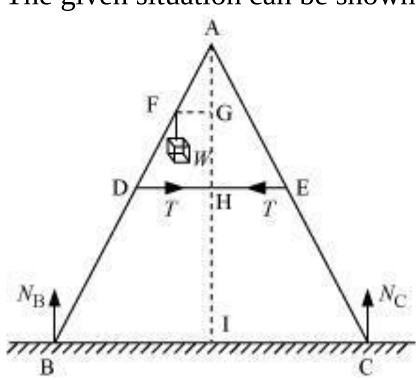
As shown in Fig.7.40, the two sides of a step ladder BA and CA are 1.6 m long and hinged at A. A rope DE, 0.5 m is tied half way up. A weight 40 kg is suspended from a point F, 1.2 m from B along the ladder BA. Assuming the floor to be frictionless and neglecting the weight of the ladder, find the tension in the rope and forces exerted by the floor on the ladder. (Take $g = 9.8\ m/s^2$)

(Hint: Consider the equilibrium of each side of the ladder separately.)



Answer

The given situation can be shown as:



N_B = Force exerted on the ladder by the floor point B

N_C = Force exerted on the ladder by the floor point C

T = Tension in the rope

BA = CA = 1.6 m

DE = 0. 5 m

BF = 1.2 m

Mass of the weight, m = 40 kg

Draw a perpendicular from A on the floor BC. This intersects DE at mid-point H.

ΔABI and ΔAIC are similar

∴ BI = IC

Hence, I is the mid-point of BC.

DE || BC

BC = 2 × DE = 1 m AF =

BA – BF = 0.4 m … (i)

D is the mid-point of AB.

Hence, we can write:

$$\text{AD} = \frac{1}{2} \times \text{BA} = 0.8\,\text{m} \qquad \ldots(ii)$$

Using equations (i) and (ii), we get:

FE = 0.4 m

Hence, F is the mid-point of AD.

FG||DH and F is the mid-point of AD. Hence, G will also be the mid-point of AH.

ΔAFG and ΔADH are similar

$$\therefore \frac{FG}{DH} = \frac{AF}{AD}$$

$$\frac{FG}{DH} = \frac{0.4}{0.8} = \frac{1}{2}$$

$$FG = \frac{1}{2}DH$$

$$= \frac{1}{2} \times 0.25 = 0.125 \text{ m}$$

In ΔADH:

$$AH = \sqrt{AD^2 - DH^2}$$
$$= \sqrt{(0.8)^2 - (0.25)^2} = 0.76 \text{ m}$$

For translational equilibrium of the ladder, the upward force should be equal to the downward force.

$N_c + N_B = mg = 392$ … (*iii*)

For rotational equilibrium of the ladder, the net moment about A is:

$$-N_B \times BI + mg \times FG + N_C \times CI + T \times AG - T \times AG = 0$$
$$-N_B \times 0.5 + 40 \times 9.8 \times 0.125 + N_C \times (0.5) = 0$$
$$(N_C - N_B) \times 0.5 = 49$$
$$N_C - N_B = 98 \quad \text{... } (iv)$$

Adding equations (*iii*) and (*iv*), we get:

$N_C = 245$ N

$N_B = 147$ N

For rotational equilibrium of the side AB, consider the moment about A.

$$-N_B \times \text{BI} + mg \times \text{FG} + T \times \text{AG} = 0$$
$$-245 \times 0.5 + 40 + 9.8 \times 0.125 + T \times 0.76 = 0$$
$$0.76T = 122.5 - 49$$
$$\therefore T = 96.7 \text{ N}$$

Question 7.23:

A man stands on a rotating platform, with his arms stretched horizontally holding a 5 kg weight in each hand. The angular speed of the platform is 30 revolutions per minute. The man then brings his arms close to his body with the distance of each weight from the axis changing from 90cm to 20cm. The moment of inertia of the man together with the platform may be taken to be constant and equal to 7.6 kg m^2.

What is his new angular speed? (Neglect friction.)

Is kinetic energy conserved in the process? If not, from where does the change come about?

Answer

58.88 rev/min **(b)** No

(a) Moment of inertia of the man-platform system = 7.6 kg m^2

Moment of inertia when the man stretches his hands to a distance of 90 cm:

$2 \times m\, r^2$

$= 2 \times 5 \times (0.9)^2$

= 8.1 kg m^2

Initial moment of inertia of the system, $I_i = 7.6 + 8.1 = 15.7 \text{ kg m}^2$

Angular speed, $\omega_i = 300 \text{ rev/min}$

Angular momentum, $L_i = I_i\omega_i = 15.7 \times 30$ ... (i)

Moment of inertia when the man folds his hands to a distance of 20 cm:

2 × mr^2

= 2 × 5 $(0.2)^2$ = 0.4 kg m^2

Final moment of inertia, $I_f = 7.6 + 0.4 = 8 \text{ kg m}^2$

Final angular speed = ω_f

Final angular momentum, $L_f = I_f\omega_f = 0.79\omega_f$ … (ii)

From the conservation of angular momentum, we have:

$I_i\omega_i = I_f\omega_f$

$\therefore \omega_f = \dfrac{15.7 \times 30}{8} = 58.88 \text{ rev/min}$

(b) Kinetic energy is not conserved in the given process. In fact, with the decrease in the moment of inertia, kinetic energy increases. The additional kinetic energy comes from the work done by the man to fold his hands toward himself.

Question 7.24:

A bullet of mass 10 g and speed 500 m/s is fired into a door and gets embedded exactly at the centre of the door. The door is 1.0 m wide and weighs 12 kg. It is hinged at one end and rotates about a vertical axis practically without friction. Find the angular speed of the door just after the bullet embeds into it.

(Hint: The moment of inertia of the door about the vertical axis at one end is $ML^2/3$.)

Answer

Mass of the bullet, m = 10 g = 10×10^{-3} kg

Velocity of the bullet, v = 500 m/s

Thickness of the door, L = 1 m

Radius of the door, $r = \frac{1}{2}$ m

Mass of the door, M = 12 kg

Angular momentum imparted by the bullet on the door:

$\alpha = mvr$

$$= \left(10 \times 10^{-3}\right) \times (500) \times \frac{1}{2} = 2.5 \text{ kg m}^2\text{s}^{-1} \quad \ldots (i)$$

Moment of inertia of the door:

$$I = \frac{1}{3} ML^2$$

$$= \frac{1}{3} \times 12 \times (1)^2 = 4 \text{kg m}^2$$

But $\alpha = I\omega$

$$\therefore \omega = \frac{\alpha}{I}$$

$$= \frac{2.5}{4} = 0.625 \text{ rad s}^{-1}$$

Question 7.25:
Two discs of moments of inertia I_1 and I_2 about their respective axes (normal to the disc and passing through the centre), and rotating with angular speeds ω_1 and ω_2 are brought into contact face to face with their axes of rotation coincident. (a) What is the angular speed of the two-disc system? (b) Show that the kinetic energy of the combined system is less than the sum of the initial kinetic energies of the two discs. How do you account for this loss in energy? Take $\omega_1 \neq \omega_2$.

Answer
(a)

Moment of inertia of disc I = I_1
Angular speed of disc I = ω_1

Angular speed of disc II = I_2
Angular momentum of disc II = ω_1

Angular momentum of disc I, $L_1 = I_1\omega_1$
Angular momentum of disc II, $L_2 = I_2\omega_2$

Total initial angular momentum, $L_i = I_1\omega_1 + I_2\omega_2$

When the two discs are joined together, their moments of inertia get added up.

Moment of inertia of the system of two discs, $I = I_1 + I_2$

Let ω be the angular speed of the system.

Total final angular momentum, $L_f = (I_1 + I_2)\omega$

Using the law of conservation of angular momentum, we have:

$L_i = L_f$

$I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega$

$$\therefore \omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$$

(b) Kinetic energy of disc I, $E_1 = \frac{1}{2} I_1\omega_1^2$

Kinetic energy of disc II, $E_2 = \frac{1}{2} I_2\omega_2^2$

Total initial kinetic energy, $E_i = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2\right)$

When the discs are joined, their moments of inertia get added up.

Moment of inertia of the system, $I = I_1 + I_2$

Angular speed of the system = ω

Final kinetic energy E_f:

$$= \frac{1}{2}(I_1 + I_2)\omega^2$$

$$= \frac{1}{2}(I_1 + I_2)\left(\frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}\right)^2 = \frac{1}{2}\frac{(I_1\omega_1 + I_2\omega_2)^2}{I_1 + I_2}$$

$\therefore E_i - E_f$

$$= \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2\right) - \frac{\left(I_1\omega_1 + I_2\omega_2\right)^2}{2\left(I_1 + I_2\right)}$$

$$= \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 - \frac{1}{2}\frac{I_1^2\omega_1^2}{\left(I_1 + I_2\right)} - \frac{1}{2}\frac{I_2^2\omega_2^2}{\left(I_1 + I_2\right)} - \frac{1}{2}\frac{2I_1I_2\omega_1\omega_2}{\left(I_1 + I_2\right)}$$

$$= \frac{1}{\left(I_1 + I_2\right)}\left[\frac{1}{2}I_1^2\omega_1^2 + \frac{1}{2}I_1I_2\omega_1^2 + \frac{1}{2}I_1I_2\omega_2^2 + \frac{1}{2}I_2^2\omega^2 - \frac{1}{2}I_1^2\omega_1^2 - \frac{1}{2}I_2^2\omega_2^2 - I_1I_2\omega_1\omega_2\right]$$

$$= \frac{I_1I_2}{2\left(I_1 + I_2\right)}\left[\omega_1^2 + \omega_2^2 - 2\omega_1\omega_2\right]$$

$$= \frac{I_1I_2\left(\omega_1 - \omega_2\right)^2}{2\left(I_1 + I_2\right)}$$

All the qauantities on RHS are positive.

$\therefore E_i - E_f > 0$

$E_i > E_f$

The loss of KE can be attributed to the frictional force that comes into play when the two discs come in contact with each other.

Question 7.26:

Prove the theorem of perpendicular axes.

(Hint: Square of the distance of a point (x, y) in the x–y plane from an axis through the origin perpendicular to the plane is $x^2 + y^2$).

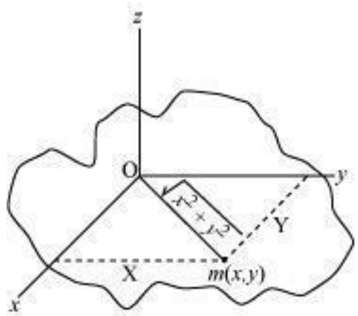
Prove the theorem of parallel axes.

(Hint: If the centre of mass is chosen to be the origin $\sum m_i \mathbf{r}_i = 0$).

Answer

(a) The theorem of perpendicular axes states that the moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.

A physical body with centre O and a point mass *m*,in the *x*–*y* plane at (*x*, *y*) is shown in the following figure.



Moment of inertia about *x*-axis, $I_x = mx^2$

Moment of inertia about *y*-axis, $I_y = my^2$

Moment of inertia about *z*-axis, $I_z = m\left(\sqrt{x^2+y^2}\right)^2$

$I_x + I_y = mx^2 + my^2$

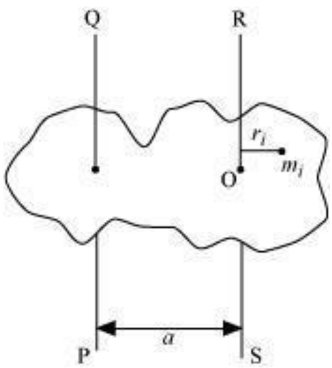
$= m(x^2 + y^2)$

$= m\left(\sqrt{x^2+y^2}\right)^2$

$I_x + I_y = I_z$

Hence, the theorem is proved.

(b) The theorem of parallel axes states that the moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.



Suppose a rigid body is made up of n particles, having masses m_1, m_2, m_3, … , m_n, at perpendicular distances r_1, r_2, r_3, … , r_n respectively from the centre of mass O of the rigid body.

The moment of inertia about axis RS passing through the point O:

$$I_{RS} = \sum_{i=1}^{n} m_i r_i^2$$

The perpendicular distance of mass m_i, from the axis QP = $a + r_i$

Hence, the moment of inertia about axis QP:

$$I_{QP} = \sum_{i=1}^{n} m_i (a + r_i)^2$$
$$= \sum_{i=1}^{n} m_i (a^2 + r_i^2 + 2ar_i)^2$$
$$= \sum_{i=1}^{n} m_i a^2 + \sum_{i=1}^{n} m_i r_i^2 + \sum_{i=1}^{n} m_i 2ar_i$$
$$= I_{RS} + \sum_{i=1}^{n} m_i a^2 + 2\sum_{i=1}^{n} m_i ar_i^2$$

Now, at the centre of mass, the moment of inertia of all the particles about the axis passing through the centre of mass is zero, that is,

$$2\sum_{i=1}^{n} m_i ar_i = 0$$
$$\because a \neq 0$$
$$\therefore \sum m_i r_i = 0$$

Also,

$$\sum_{i=1}^{n} m_i = M; \quad M = \text{Total mass of the rigid body}$$
$$\therefore I_{QP} = I_{RS} + Ma^2$$

Hence, the theorem is proved.

Question 7.27:

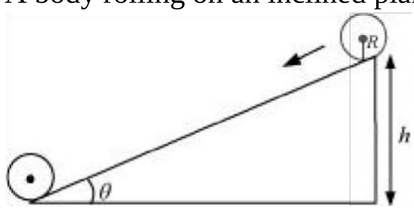
Prove the result that the velocity *v* of translation of a rolling body (like a ring, disc, cylinder or sphere) at the bottom of an inclined plane of a height *h* is given by

$$v^2 = \frac{2gh}{(1 + k^2 / R^2)}.$$

Using dynamical consideration (i.e. by consideration of forces and torques). Note k is the radius of gyration of the body about its symmetry axis, and R is the radius of the body. The body starts from rest at the top of the plane.

Answer

A body rolling on an inclined plane of height h,is shown in the following figure:



m = Mass of the body

R = Radius of the body

K = Radius of gyration of the body

v = Translational velocity of the

body h =Height of the inclined

plane g = Acceleration due to

gravity

Total energy at the top of the plane, $E_1 = mgh$

Total energy at the bottom of the plane, $E_b = KE_{rot} + KE_{trans}$

$$= \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$$

But $I = mk^2$ and $\omega = \frac{v}{R}$

$$\therefore E_b = \frac{1}{2}\left(mk^2\right)\left(\frac{v^2}{R^2}\right) + \frac{1}{2}mv^2$$

$$= \frac{1}{2}mv^2\frac{k^2}{R^2} + \frac{1}{2}mv^2$$

$$= \frac{1}{2}mv^2\left(1 + \frac{k^2}{R^2}\right)$$

From the law of conservation of energy, we have:

$$E_T = E_b$$

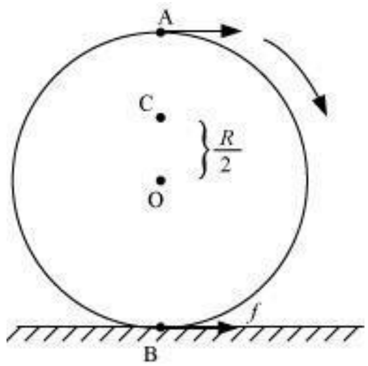
$$mgh = \frac{1}{2}mv^2\left(1 + \frac{k^2}{R^2}\right)$$

$$\therefore v = \frac{2gh}{\left(1 + k^2 / R^2\right)}$$

Hence, the given result is proved.

Question 7.28:

A disc rotating about its axis with angular speed $\omega_o$is placed lightly (without any translational push) on a perfectly frictionless table. The radius of the disc is *R*. What are the linear velocities of the points A, B and C on the disc shown in Fig. 7.41? Will the disc roll in the direction indicated?



Answer $v_A = R\omega_o$; $v_B = R\omega_o$; $v_c = \left(\frac{R}{2}\right)\omega_o$; The disc will not roll
Angular speed of the disc = ω_o

Radius of the disc = R

Using the relation for linear velocity, $v = \omega_o R$

For point A:

$v_A = R\omega_o$; in the direction tangential to the right

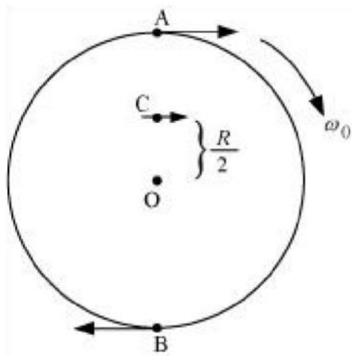
For point B:

$v_B = R\omega_o$; in the direction tangential to the left

For point C:

$v_c = \left(\frac{R}{2}\right)\omega_o$; in the direction same as that of v_A

The directions of motion of points A, B, and C on the disc are shown in the following figure



Since the disc is placed on a frictionless table, it will not roll. This is because the presence of friction is essential for the rolling of a body.

Question 7.29:

Explain why friction is necessary to make the disc in Fig. 7.41 roll in the direction indicated.

Give the direction of frictional force at B, and the sense of frictional torque, before perfect rolling begins.

What is the force of friction after perfect rolling begins?

Answer

A torque is required to roll the given disc. As per the definition of torque, the rotating force should be tangential to the disc. Since the frictional force at point B is along the tangential force at point A, a frictional force is required for making the disc roll.

Force of friction acts opposite to the direction of velocity at point B. The direction of linear velocity at point B is tangentially leftward. Hence, frictional force will act tangentially rightward. The sense of frictional torque before the start of perfect rolling is perpendicular to the plane of the disc in the outward direction.

Since frictional force acts opposite to the direction of velocity at point B, perfect rolling will begin when the velocity at that point becomes equal to zero. This will make the frictional force acting on the disc zero.

Question 7.30:

A solid disc and a ring, both of radius 10 cm are placed on a horizontal table simultaneously, with initial angular speed equal to 10 π rad s^{-1}. Which of the two will start to roll earlier? The co-efficient of kinetic friction is $\mu_k = 0.2$.

Answer

Disc

Radii of the ring and the disc, r = 10 cm = 0.1 m

Initial angular speed, $\omega_0 = 10\ \pi$ rad s^{-1}

Coefficient of kinetic friction, $\mu_k = 0.2$

Initial velocity of both the objects, $u = 0$

Motion of the two objects is caused by frictional force. As per Newton's second law of motion, we have frictional force, $f = ma$ $\mu_k mg = ma$ Where, a = Acceleration produced in the objects m = Mass $\therefore a = \mu_k g$ … (i)

As per the first equation of motion, the final velocity of the objects can be obtained as:

$v = u + at$

$= 0 + \mu_k g t$

$= \mu_k g t$ … (ii)

The torque applied by the frictional force will act in perpendicularly outward direction and cause reduction in the initial angular speed.

Torque, $\tau = -I\alpha$
α = Angular acceleration

$\mu_x mgr = -I\alpha$

$$\therefore \alpha = \frac{-\mu_k mgr}{I} \quad \ldots (iii)$$

Using the first equation of rotational motion to obtain the final angular speed:

$$\omega = \omega_0 + \alpha t$$
$$= \omega_0 + \frac{-\mu_k mgr}{I} t \quad \ldots (iv)$$

Rolling starts when linear velocity, $v = r\omega$

$$\therefore v = r\left(\omega_0 - \frac{\mu_k gmrt}{I}\right) \quad \ldots (v)$$

Equating equations (*ii*) and (*v*), we get:

$$\mu_k gt = r\left(\omega_0 - \frac{\mu_k gmrt}{I}\right)$$

$$= r\omega_0 - \frac{\mu_k gmr^2 t}{I} \quad \ldots (vi)$$

For the ring: $I = mr^2$

$$\therefore \mu_k g t = r\omega_0 - \frac{\mu_k g m r^2 t}{mr^2}$$

$$= r\omega_0 - \mu_k g m t_r$$

$$2\mu_k g t = r\omega_0$$

$$\therefore t_r = \frac{r\omega_0}{2\mu_k g}$$

$$= \frac{0.1 \times 10 \times 3.14}{2 \times 0.2 \times 9.8} = 0.80 \text{ s} \qquad \ldots (vii)$$

For the disc: $I = \frac{1}{2}mr^2$

$$\therefore \mu_k g t_d = r\omega_0 \frac{\mu_k g m r^2 t}{\frac{1}{2}mr^2}$$

$$= r\omega_0 - 2\mu_k g t$$

$$3\mu_k g t_d = r\omega_0$$

$$\therefore t_d = \frac{r\omega_0}{3\mu_k g}$$

$$= \frac{0.1 \times 10 \times 3.14}{3 \times 0.2 \times 9.8} = 0.53 \text{ s} \qquad \ldots (viii)$$

Since $t_d > t_r$, the disc will start rolling before the ring.

Question 7.31:

A cylinder of mass 10 kg and radius 15 cm is rolling perfectly on a plane of inclination 30°. The coefficient of static friction $\mu_s = 0.25$.

How much is the force of friction acting on the cylinder?

What is the work done against friction during rolling?

If the inclination θ of the plane is increased, at what value of θ does the cylinder begin to skid, and not roll perfectly?

Answer

Mass of the cylinder, $m = 10$ kg

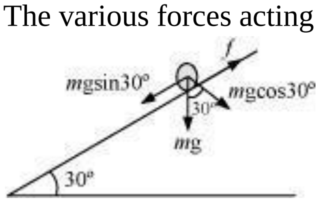
Radius of the cylinder, $r = 15$ cm $= 0.15$ m

Co-efficient of kinetic friction, $\mu_k = 0.25$

Angle of inclination, $\theta = 30°$

Moment of inertia of a solid cylinder about its geometric axis, $I = \frac{1}{2}mr^2$

The various forces acting on the cylinder are shown in the following figure:



The acceleration of the cylinder is given as:

$$a = \frac{mg\sin\theta}{m + \frac{I}{r^2}}$$

$$= \frac{mg\sin\theta}{m + \frac{1}{2}\frac{mr^2}{r^2}} = \frac{2}{3}g\sin 30^\circ$$

$$= \frac{2}{3} \times 9.8 \times 0.5 = 3.27 \text{ m/s}^2$$

Using Newton's second law of motion, we can write net force as:

$f_{\text{net}} = ma$

$$mg\sin 30^\circ - f = ma$$
$$f = mg\sin 30^\circ - ma$$
$$= 10 \times 9.8 \times 0.5 - 10 \times 3.27$$
$$= 49 - 32.7 = 16.3 \text{ N}$$

During rolling, the instantaneous point of contact with the plane comes to rest. Hence, the work done against frictional force is zero.

For rolling without skid, we have the relation:

$$\mu = \frac{1}{3}\tan\theta$$

$$\tan\theta = 3\mu = 3 \times 0.25$$
$$\therefore \theta = \tan^{-1}(0.75) = 36.87^\circ$$

Question 7.32:

Read each statement below carefully, and state, with reasons, if it is true or false;

During rolling, the force of friction acts in the same direction as the direction of motion of the CM of the body.

The instantaneous speed of the point of contact during rolling is zero.

The instantaneous acceleration of the point of contact during rolling is zero.

For perfect rolling motion, work done against friction is zero.

A wheel moving down a perfectly *frictionless* inclined plane will undergo slipping (not rolling) motion.

Answer

False

Frictional force acts opposite to the direction of motion of the centre of mass of a body. In the case of rolling, the direction of motion of the centre of mass is backward. Hence, frictional force acts in the forward direction.

True

Rolling can be considered as the rotation of a body about an axis passing through the point of contact of the body with the ground. Hence, its instantaneous speed is zero.

False

When a body is rolling, its instantaneous acceleration is not equal to zero. It has some value.

True

When perfect rolling begins, the frictional force acting at the lowermost point becomes zero. Hence, the work done against friction is also zero.

True

The rolling of a body occurs when a frictional force acts between the body and the surface. This frictional force provides the torque necessary for rolling. In the absence of a frictional force, the body slips from the inclined plane under the effect of its own weight.

Question 7.33:

Separation of Motion of a system of particles into motion of the centre of mass and motion about the centre of mass:

Show $\mathbf{p}_i = \mathbf{p}'_i + m_i\mathbf{V}$

Where $\mathbf{p}_i$ is the momentum of the i^{th} particle (of mass m_i) and $\mathbf{p}'_i = m_i \mathbf{v}'_i$. Note $\mathbf{v}'_i$ is the velocity of the i^{th} particle relative to the centre of mass.

Also, prove using the definition of the centre of mass $\sum_i \mathbf{p}'_i = 0$

Show $K = K' + \frac{1}{2}MV^2$

Where K is the total kinetic energy of the system of particles, K' is the total kinetic energy of the system when the particle velocities are taken with respect to the centre of mass and $MV^2/2$ is the kinetic energy of the translation of the system as a whole (i.e. of the centre of mass motion of the system). The result has been used in Sec. 7.14.

Show $\mathbf{L} = \mathbf{L}' + \mathbf{R} \times M\mathbf{V}$

Where $\mathbf{L}' = \sum_i \mathbf{r}'_i \times \mathbf{p}'_i$ is the angular momentum of the system about the centre of mass with velocities taken relative to the centre of mass. Remember $\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R}$; rest of the notation is the standard notation used in the chapter. Note $\mathbf{L}'$ and $M\mathbf{R} \times \mathbf{V}$ can be said to be angular momenta, respectively, about and of the centre of mass of the system of particles.

Show $\frac{d\mathbf{L}'}{dt} = \sum_i \mathbf{r}'_i \times \frac{d}{dt}(\mathbf{p}'_i)$

Further, show that

$$\frac{d\mathbf{L}'}{dt} = \tau'_{\text{ext}}$$

where τ'_{ext} is the sum of all external torques acting on the system about the centre of mass.

(Hint: Use the definition of centre of mass and Newton's Third Law. Assume the internal forces between any two particles act along the line joining the particles.)

Answer

(a) Take a system of i moving particles.

Mass of the i^{th} particle = m_i

Velocity of the i^{th} particle = $\mathbf{v}_i$

Hence, momentum of the i^{th} particle, $\mathbf{p}_i = m_i\,\mathbf{v}_i$

Velocity of the centre of mass = $\mathbf{V}$

The velocity of the i^{th} particle with respect to the centre of mass of the system is given

as: $\mathbf{v}'_i = \mathbf{v}_i - \mathbf{V}$ … (1)

Multiplying m_i throughout equation (1), we get:

$m_i\,\mathbf{v}'_i = m_i\,\mathbf{v}_i - m_i\,\mathbf{V}$

$\mathbf{p}'_i = \mathbf{p}_i - m_i\,\mathbf{V}$

Where,

$\mathbf{p}_i' = m_i\mathbf{v}_i'$ = Momentum of the i^{th} particle with respect to the centre of mass of the system $\therefore \mathbf{p}_i = \mathbf{p}'_i + m_i \mathbf{V}$

We have the relation: $\mathbf{p}'_i = m_i\mathbf{v}_i'$

Taking the summation of momentum of all the particles with respect to the centre of mass of the system, we get:

$$\sum_i \mathbf{p}'_i = \sum_i m_i \mathbf{v}'_i = \sum_i m_i \frac{d\mathbf{r}'_i}{dt}$$

Where,

$\mathbf{r}'_i$ = Position vector of ith particle with respect to the centre of mass

$$\mathbf{v}'_i = \frac{d\mathbf{r}'_i}{dt}$$

As per the definition of the centre of mass, we have:

$$\sum_i m_i \mathbf{r}'_i = 0$$

$$\therefore \sum_i m_i \frac{d\mathbf{r}'_i}{dt} = 0$$

$$\sum_i \mathbf{p}'_i = 0$$

We have the relation for velocity of the i^{th} particle as:

$\mathbf{v}_i = \mathbf{v}'_i + \mathbf{V}$

$$\sum_i m_i \mathbf{v}_i = \sum_i m_i \mathbf{v}'_i + \sum_i m_i \mathrm{V} \quad \ldots (2)$$

Taking the dot product of equation (2) with itself, we get:

$$\sum_i m_i \mathbf{v}_i . \sum_i m_i \mathbf{v}_i = \sum_i m_i (\mathbf{v}'_i + \mathbf{V}) . \sum_i m_i (\mathbf{v}'_i + \mathbf{V})$$

$$M^2 \sum_i v_i^2 = M^2 \sum_i v_i'^2 + M^2 \sum_i \mathbf{v}_i . \mathbf{v}'_i + M^2 \sum_i \mathbf{v}'_i . \mathbf{v}_i + M^2 V^2$$

Here, for the centre of mass of the system of particles, $\sum_i \mathbf{v}_i . \mathbf{v}'_i = -\sum_i \mathbf{v}'_i . \mathbf{v}_i$

$$M^2 \sum_i v_i^2 = M^2 \sum_i v_i'^2 + M^2 V^2$$

$$\frac{1}{2} M \sum_i v_i^2 = \frac{1}{2} M \sum_i v_i'^2 + \frac{1}{2} M V^2$$

$$K = K' + \frac{1}{2} M V^2$$

Where,

$K = \frac{1}{2} M \sum_i v_i^2$ = Total kinetic energy of the system of particles

$K' = \frac{1}{2} M \sum_i v_i'^2$ = Total kinetic energy of the system of particles with respect to the centre of mass

$\frac{1}{2} M V^2$ = Kinetic energy of the translation of the system as a whole

Position vector of the i^{th} particle with respect to origin = $\mathbf{r}_i$

Position vector of the i^{th} particle with respect to the centre of mass = $\mathbf{r}'_i$

Position vector of the centre of mass with respect to the origin = $\mathbf{R}$

It is given that: $\mathbf{r}'_i = \mathbf{r}_i$

$-\ \mathbf{R}\ \mathbf{r}_i = \mathbf{r}'_i + \mathbf{R}$ We

have from part **(a)**, $\mathbf{p}_i$

$= \mathbf{p}'_i + m_i \mathbf{V}$

Taking the cross product of this relation by $\mathbf{r}_i$, we get:

$$\sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i \mathbf{r}_i \times \mathbf{p}'_i + \sum_i \mathbf{r}_i \times m_i \mathbf{V}$$

$$\mathbf{L} = \sum_i (\mathbf{r}'_i + \mathbf{R}) \times \mathbf{p}'_i + \sum_i (\mathbf{r}'_i + \mathbf{R}) \times m_i \mathbf{V}$$

$$= \sum_i \mathbf{r}'_i \times \mathbf{p}'_i + \sum_i \mathbf{R} \times \mathbf{p}'_i + \sum_i \mathbf{r}'_i \times m_i \mathbf{V} + \sum_i \mathbf{R} \times m_i \mathbf{V}$$

$$= \mathbf{L}' + \sum_i \mathbf{R} \times \mathbf{p}'_i + \sum_i \mathbf{r}'_i \times m_i \mathbf{V} + \sum_i \mathbf{R} \times m_i \mathbf{V}$$

Where,

$$\mathbf{R} \times \sum_i \mathbf{p}'_i = 0 \quad \text{and}$$

$$\left(\sum_i \mathbf{r}'_i\right) \times M\mathbf{V} = \mathbf{0}$$

$$\sum_i m_i = M$$

$$\therefore \mathbf{L} = \mathbf{L}' + R \times M\mathbf{V}$$

We have the relation:

$$\mathbf{L}' = \sum_i \mathbf{r}'_i \times \mathbf{p}'_i$$

$$\frac{d\mathbf{L}'}{dt} = \frac{d}{dt}\left(\sum_i \mathbf{r}'_i \times \mathbf{p}'_i\right)$$

$$= \frac{d}{dt}\left(\sum_i \mathbf{r}'_i\right) \times \mathbf{p}'_i + \sum_i \mathbf{r}'_i \times \frac{d}{dt}(\mathbf{p}'_i)$$

$$= \frac{d}{dt}\left(\sum_i m_i \mathbf{r}'_i\right) \times \mathbf{v}'_i + \sum_i \mathbf{r}'_i \times \frac{d}{dt}(\mathbf{p}'_i)$$

Where, $\mathbf{r}'_i$ is the position vector with respect to the centre of mass of the system of particles.

$$\therefore \sum_i m_i \mathbf{r}'_i = 0$$

$$\therefore \frac{d\mathbf{L}'}{dt} = \sum_i \mathbf{r}'_i \times \frac{d}{dt}(\mathbf{p}'_i)$$

We have the relation:

$$\frac{d\mathbf{L}'}{dt} = \sum_i \mathbf{r}'_i \times \frac{d}{dt}(\mathbf{p}'_i)$$

$$= \sum_i \mathbf{r}'_i \times m_i \frac{d}{dt}(\mathbf{v}'_i)$$

Where, $\frac{d}{dt}(\mathbf{v}'_i)$ is the rate of change of velocity of the ith particle with respect ot the centre of mass of the system

Therefore, according to Newton's third law of motion, we can write:

$$m_i \frac{d}{dt}(\mathbf{v}'_i) = \text{Extrenal force acting on the } i\text{th particle} = \sum_i (\tau'_i)_{\text{ext}}$$

i.e., $\sum_i \mathbf{r}'_i \times m_i \frac{d}{dt}(\mathbf{v}'_i) = \tau'_{\text{ext}}$ = External torque acting on the system as a whole

$$\therefore \frac{d\mathbf{L}'}{dt} = \tau'_{\text{ext}}$$