Chapter 1

Relations and Functions

- Mathematics XII
- Exercise 1.2

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Exercise 1.2

Question 1:

Show that the function f: $\mathbf{R}_* \rightarrow \mathbf{R}_*$ defined by $f(x) = \frac{1}{x}$ is one-one and onto, where $\mathbf{R}_*$ is the set of all non-zero real numbers. Is the result true, if the domain $\mathbf{R}_*$ is replaced by **N** with co-domain being same as $\mathbf{R}_*$?

Answer 1:

It is given that f: $R_* \rightarrow R_*$ is defined by $f(x) = \frac{1}{x}$

For one – one:

Let $x, y \in R_*$ such that $f(x) = f(y)$

$\Rightarrow \frac{1}{x} = \frac{1}{y}$

$\Rightarrow x = y$

$\therefore f$ is one – one.

For onto:

It is clear that for $y \in R_*$, there exists $x = \frac{1}{y} \in R_*$ [as $y \neq 0$] such that

$$f(x) = \frac{1}{\left(\frac{1}{y}\right)} = y$$

$\therefore f$ is onto.

Thus, the given function f is one – one and onto.

Now, consider function g: $\mathbf{N} \rightarrow R_*$ defined by $g(x) = \frac{1}{x}$

We have,

$g(x_1) = g(x_2)$ $\Rightarrow \frac{1}{x_1} = \frac{1}{x_2}$ $\Rightarrow x_1 = x_2$

∴ g is one – one.

Further, it is clear that g is not onto as for $1.2 \in R_*$ there does not exit any x in **N** such that $g(x) = \frac{1}{1.2}$.

Hence, function g is one-one but not onto.

Question 2:

Check the injectivity and surjectivity of the following functions:

(i) f: **N** → **N** given by $f(x) = x^2$

(ii) f: **Z** → **Z** given by $f(x) = x^2$

(iii) f: **R** → **R** given by $f(x) = x^2$

(iv) f: **N** → **N** given by $f(x) = x^3$

(v) f: **Z** → **Z** given by $f(x) = x^3$

Answer 2:

(i) f: **N** → **N** is given by $f(x) = x^2$

It is seen that for $x, y \in$ **N**, $f(x) = f(y) \Rightarrow x^2 = y^2 \Rightarrow x = y$.

∴ f is injective.

Now, $2 \in$ **N**. But, there does not exist any x in **N** such that $f(x) = x^2 = 2$.

∴ f is not surjective.

Hence, function f is injective but not surjective.

(ii) f: **Z** → **Z** is given by $f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

∴ f is not injective.

Now, $-2 \in$ **Z**. But, there does not exist any element $x \in$ **Z** such that

$f(x) = -2$ or $x^2 = -2$.

$\therefore f$ is not surjective.

Hence, function f is neither injective nor surjective.

(iii) f: **R** → **R** is given by $f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

$\therefore f$ is not injective.

Now, $-2 \in$ **R**. But, there does not exist any element $x \in$ **R** such that

$f(x) = -2$ or $x^2 = -2$.

$\therefore f$ is not surjective.

Hence, function f is neither injective nor surjective.

(iv) f: **N** → **N** given by $f(x) = x^3$

It is seen that for $x, y \in$ **N**, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in$ **N**. But, there does not exist any element $x \in$ **N** such that

$f(x) = 2$ or $x^3 = 2$.

$\therefore f$ is not surjective

Hence, function f is injective but not surjective.

(v) f: **Z** → **Z** is given by $f(x) = x^3$

It is seen that for $x, y \in$ **Z**, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in$ **Z**. But, there does not exist any element $x \in$ **Z** such that

$f(x) = 2$ or $x^3 = 2$.

$\therefore f$ is not surjective.

Hence, function f is injective but not surjective.

Question 3:

Prove that the Greatest Integer Function $f: \mathbf{R} \to \mathbf{R}$ given by $f(x) = [x]$, is neither one – one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

Answer 3:

$f: \mathbf{R} \to \mathbf{R}$ is given by, $f(x) = [x]$

It is seen that $f(1.2) = [1.2] = 1$, $f(1.9) = [1.9] = 1$.

$\therefore f(1.2) = f(1.9)$, but $1.2 \neq 1.9$.

$\therefore f$ is not one – one.

Now, consider $0.7 \in \mathbf{R}$.

It is known that $f(x) = [x]$ is always an integer. Thus, there does not exist any element $x \in \mathbf{R}$ such that $f(x) = 0.7$.

$\therefore f$ is not onto.

Hence, the greatest integer function is neither one – one nor onto.

Question 4:

Show that the Modulus Function $f: \mathbf{R} \to \mathbf{R}$ given by $f(x) = |x|$, is neither one – one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

Answer 4:

$f: \mathbf{R} \to \mathbf{R}$ is given by $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

It is clear that $f(-1) = |-1| = 1$ and $f(1) = |1| = 1$

$\therefore f(-1) = f(1)$, but $-1 \neq 1$.

$\therefore f$ is not one – one.

Now, consider $-1 \in \mathbf{R}$.

It is known that $f(x) = |x|$ is always non-negative. Thus, there does not exist any element x in domain $\mathbf{R}$ such that $f(x) = |x| = -1$.

$\therefore f$ is not onto.

Hence, the modulus function is neither one-one nor onto.

Question 5:

Show that the Signum Function f: **R** → **R**, given by $f(x) = \begin{cases} 1, & if\ x > 0 \\ 0, & if\ x = 0 \\ -1, & if\ x < 0 \end{cases}$

is neither one-one nor onto.

Answer 5:

f: **R** → **R** is given by $f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$

It is seen that $f(1) = f(2) = 1$, but $1 \neq 2$.

$\therefore f$ is not one – one.

Now, as $f(x)$ takes only 3 values (1, 0, or −1) for the element −2 in co-domain **R**, there does not exist any x in domain **R** such that $f(x) = -2$.

$\therefore f$ is not onto.

Hence, the Signum function is neither one – one nor onto.

Question 6:

Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Show that f is one – one.

Answer 6:

It is given that $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$.

$f: A \to B$ is defined as $f = \{(1, 4), (2, 5), (3, 6)\}$.

$\therefore f(1) = 4,\quad f(2) = 5,\quad f(3) = 6$

It is seen that the images of distinct elements of A under f are distinct.

Hence, function f is one – one.

Question 7:

In each of the following cases, state whether the function is one – one, onto or bijective.

Justify your answer.

(i) $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3 - 4x$

(ii) $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1 + x^2$

Answer 7:

(i) $f: \mathbf{R} \to \mathbf{R}$ is defined as $f(x) = 3 - 4x$.

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow 3\text{-}4x_1 = 3\text{-}4x_2$

$\Rightarrow -4x_1 = -4x_2$

$\Rightarrow x_1 = x_2$

$\therefore f$ is one – one.

For any real number (y) in **R**, there exists $\frac{3-y}{4}$ in **R** such that

$$f\left(\frac{3-y}{4}\right) = 3\text{-}4\left(\frac{3-y}{4}\right) = y$$

$\therefore f$ is onto.

Hence, f is bijective.

(ii) $f: \mathbf{R} \to \mathbf{R}$ is defined as $f(x) = 1 + x^2$

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow 1 + x_1^2 = 1 + x_2^2$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

$\therefore f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$

For example f(1) = f(-1) = 2

∴ f is not one – one.

Consider an element −2 in co-domain **R**.

It is seen that $f(x) = 1 + x^2$ is positive for all $x \in \mathbf{R}$.

Thus, there does not exist any x in domain **R** such that $f(x) = -2$.

∴ f is not onto.

Hence, f is neither one – one nor onto.

Question 8:

Let A and B be sets. Show that $f: A \times B \to B \times A$ such that $(a, b) = (b, a)$ is bijective function.

Answer 8:

$f: A \times B \to B \times A$ is defined as $f(a, b) = (b, a)$.

Let $(a_1, b_1), (a_2, b_2) \in A \times B$ such that $f(a_1, b_1) = f(a_2, b_2)$

$\Rightarrow (b_1, a_1) = (b_2, a_2)$

$\Rightarrow b_1 = b_2$ and $a_1 = a_2$

$\Rightarrow (a_1, b_1) = (a_2, b_2)$

∴ f is one – one.

Now, let $(b, a) \in B \times A$ be any element.

Then, there exists $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$. [By definition of f]

∴ f is onto.

Hence, f is bijective.

Question 9:

Let f: **N** → **N** be defined by $f(n) = \begin{cases} \frac{n+1}{2}, \text{ if n is odd} \\ \frac{n}{2}, \text{ if n is even} \end{cases}$ for all n ∈ **N**

State whether the function f is bijective. Justify your answer.

Answer 9:

f: **N** → **N** is defined as $f(n) = \begin{cases} \frac{n+1}{2}, \text{ if n is odd} \\ \frac{n}{2}, \text{ if n is even} \end{cases}$ for all n ∈ **N**

It can be observed that:

$f(1) = \frac{1+1}{2} = 1$ and $f(2) = \frac{2}{2} = 1$ [By definition of f(n)]

$f(1) = f(2)$, where $1 \neq 2$

∴ f is not one-one.

Consider a natural number (n) in co-domain **N**.

Case **I:** n is odd

∴ $n = 2r + 1$ for some r ∈ **N.** Then, there exists $4r + 1$ ∈ **N** such that

$$f(4r+1) = \frac{4r+1+1}{2} = 2r+1$$

Case **II:** n is even

∴ $n = 2r$ for some r ∈ **N.** Then, there exists $4r$ ∈ **N** such that

$f(4r) = \frac{4r}{2} = 2r.$

∴ f is onto.

Hence, f is not a bijective function.

Question 10:

Let A = **R** − {3} and B = **R** − {1}. Consider the function *f*: A → B defined by $(x) = \left(\frac{x-2}{x-3}\right)$. Is *f* one-one and onto? Justify your answer.

Answer 10:

A = **R** − {3}, B = **R** − {1} and *f*: A → B defined by $f(x) = \left(\frac{x-2}{x-3}\right)$

Let $x, y \in$ A such that $f(x) = f(y)$

$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$

$\Rightarrow (x - 2)(y - 3) = (y - 2)(x - 3)$

$\Rightarrow xy - 3x - 2y + 6 = xy - 2x - 3y + 6$

$\Rightarrow -3x - 2y = -2x - 3y \quad \Rightarrow x = y$

∴ *f* is one-one.

Let $y \in$ B = **R** − {1}. Then, $y \neq 1$.

The function *f* is onto if there exists $x \in$ A such that $f(x) = y$.

Now, $f(x) = y$

$\Rightarrow \frac{x-2}{x-3} = y$

$\Rightarrow x - 2 = xy - 3y \quad \Rightarrow x(1 - y) = -3y + 2$

$\Rightarrow x = \frac{2-3y}{1-y} \in A \qquad [y \neq 1]$

Thus, for any $y \in$ B, there exists $\frac{2-3y}{1-y} \in$ A such that

$$f\left(\frac{2-3y}{1-y}\right) = \frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3} = \frac{2-3y-2+2y}{2-3y-3+3y} = \frac{-y}{-1} = y$$

∴ *f* is onto.

Hence, function *f* is one – one and onto.

Question 11:

Let f: $\mathbf{R} \rightarrow \mathbf{R}$ be defined as $f(x) = x^4$. Choose the correct answer.

(A) f is one-one onto
(B) f is many-one onto
(C) f is one-one but not onto
(D) f is neither one-one nor onto

Answer 11:

f: $\mathbf{R} \rightarrow \mathbf{R}$ is defined as $f(x) = x^4$.

Let $x, y \in \mathbf{R}$ such that $f(x) = f(y)$.

$\Rightarrow x^4 = y^4$

$\Rightarrow x = \pm y$

$\therefore f(x) = f(y)$ does not imply that $x = y$.

For example $f(1) = f(-1) = 1$

$\therefore f$ is not one-one.

Consider an element 2 in co-domain **R**. It is clear that there does not exist any x in domain **R** such that $f(x) = 2$.

$\therefore f$ is not onto.

Hence, function f is neither one – one nor onto.

The correct answer is D.

Question 12:

Let f: $\mathbf{R} \rightarrow \mathbf{R}$ be defined as $f(x) = 3x$. Choose the correct answer.

(A) f is one – one onto
(B) f is many – one onto
(C) f is one – one but not onto
(D) f is neither one – one nor onto

Answer 12:

f: $\mathbf{R} \rightarrow \mathbf{R}$ is defined as $f(x) = 3x$.

Let $x, y \in \mathbf{R}$ such that $f(x) = f(y)$.

$\Rightarrow 3x = 3y$

$\Rightarrow x = y$

$\therefore f$ is one-one.

Also, for any real number (y) in co-domain **R**, there exists $\frac{y}{3}$ in **R** such that $f\left(\frac{y}{3}\right) = 3\left(\frac{y}{3}\right) = y$

$\therefore f$ is onto.

Hence, function f is one – one and onto.

The correct answer is A.