Chapter 1

Relations and Functions

- Mathematics XII
- Exercise 1.3

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Exercise 1.3

Question 1:

Let f: {1, 3, 4} → {1, 2, 5} and g: {1, 2, 5} → {1, 3} be given by f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5, 1)}. Write down gof.

Answer 1:

The functions f: {1, 3, 4} → {1, 2, 5} and g: {1, 2, 5} → {1, 3} are defined as

f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5, 1)}.

$gof(1) = g[f(1)] = g(2) = 3$ [as $f(1) = 2$ and $g(2) = 3$]

$gof(3) = g[f(3)] = g(5) = 1$ [as $f(3) = 5$ and $g(5) = 1$]

$gof(4) = g[f(4)] = g(1) = 3$ [as $f(4) = 1$ and $g(1) = 3$]

$\therefore gof = \{(1, 3), (3, 1), (4, 3)\}$

Question 2:

Let f, g and h be functions from **R** to **R**. Show that

$(f + g)oh = foh + goh$

$(f.g)oh = (foh).(goh)$

Answer 2:

To prove: $(f + g)oh = foh + goh$

LHS = $[(f + g)oh](x)$

$= (f + g)[h(x)]$

$= f[h(x)] + g[h(x)]$

$= (foh)(x) + (goh)(x)$

$= \{(foh)(x) + (goh)\}(x)$ = RHS

$\therefore \{(f + g)oh\}(x) = \{(foh)(x) + (goh)\}(x)$ for all $x \in \mathbf{R}$

Hence, $(f + g)oh = foh + goh$

To Prove: $(f.g)oh = (foh).(goh)$

LHS = $[(f.g)oh](x)$

$= (f.g)[h(x)]$

$= f[h(x)] \,.\, g[h(x)]$

$= (foh)(x) \,.\, (goh)(x)$

$= \{(foh).(goh)\}(x)$ = RHS

$\therefore [(f.g)oh](x) = \{(foh).(goh)\}(x)$ for all $x \in \mathbf{R}$

Hence, $(f.g)oh = (foh).(goh)$

Question 3:

Find gof and fog, if

(i) $f(x) = |x|$ and $g(x) = |5x - 2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

Answer 3:

(i). $f(x) = |x|$ and $g(x) = |5x-2|$

$\therefore gof(x) = g(f(x)) = g(|x|) = |5|x|-2|$

$fog(x) = f(g(x)) = f(|5x-2|) = \big||5x-2|\big| = |5x-2|$

(ii). $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

$\therefore gof(x) = g(f(x)) = g(8x^3) = (8x^3)^{\frac{1}{3}} = 2x$

$fog(x) = f(g(x)) = f(x^{\frac{1}{3}}) = 8(x^{\frac{1}{3}})^3 = 8x$

Question 4:

If $f(x) = \frac{(4x+3)}{(6x-4)}, x \neq \frac{2}{3}$, show that *fof*(*x*) = *x*, for all $x \neq \frac{2}{3}$. What is the inverse of *f* ?

Answer 4:

It is given that $f(x) = \frac{(4x+3)}{(6x\text{-}4)}, x \neq \frac{2}{3}$

$$(fof)(x) = f(f(x)) = f\left(\frac{4x+3}{6x\text{-}4}\right) = \frac{4\left(\frac{4x+3}{6x\text{-}4}\right)+3}{6\left(\frac{4x+3}{6x\text{-}4}\right)\text{-}4}$$

$$= \frac{16x+12+18x\text{-}12}{24x+18\text{-}24x+16} = \frac{34x}{34}$$

$$= x$$

$\therefore$ *fof*(*x*) = *x*, for all $x \neq \frac{2}{3}$.

$\Rightarrow$ *fof* = I_x

Hence, the given function *f* is invertible and the inverse of *f* is *f* itself.

Question 5:

State with reason whether following functions have inverse

(i) *f*: {1, 2, 3, 4} → {10} with

f = {(1, 10), (2, 10), (3, 10), (4, 10)}

(ii) *g*: {5, 6, 7, 8} → {1, 2, 3, 4} with

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

(iii) *h*: {2, 3, 4, 5} → {7, 9, 11, 13} with

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

Answer 5:

(i) f: {1, 2, 3, 4} → {10} defined as f = {(1, 10), (2, 10), (3, 10), (4, 10)}

From the given definition of f, we can see that f is a many one function as

$f(1) = f(2) = f(3) = f(4) = 10$

∴ f is not one – one.

Hence, function f does not have an inverse.

(ii) g: {5, 6, 7, 8} → {1, 2, 3, 4} defined as

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

From the given definition of g, it is seen that g is a many one function as

$g(5) = g(7) = 4$.

∴ g is not one – one.

Hence, function g does not have an inverse.

(iii) h: {2, 3, 4, 5} → {7, 9, 11, 13} defined as

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

It is seen that all distinct elements of the set {2, 3, 4, 5} have distinct images under h.

∴ Function h is one – one.

Also, h is onto since for every element y of the set {7, 9, 11, 13}, there exists an element x in the set {2, 3, 4, 5}, such that $h(x) = y$.

Thus, h is a one – one and onto function.

Hence, h has an inverse.

Question 6:

Show that f: [−1, 1] → **R**, given by $f(x) = \frac{x}{(x+2)}$ is one – one. Find the inverse of the function f: [−1, 1] → Range f.

(Hint: For $y \in$ Range f, $y = f(x) = \frac{x}{(x+2)}$, for some x in $[-1, 1]$, i.e., $x = \frac{2y}{(1-y)}$)

Answer 6:

$f: [-1, 1] \rightarrow \mathbf{R}$ is given as $f(x) = \frac{x}{(x+2)}$

For one – one

Let $f(x) = f(y)$

$\Rightarrow \frac{x}{x+2} = \frac{y}{y+2}$

$\Rightarrow xy + 2x = xy + 2y$

$\Rightarrow 2x = 2y$

$\Rightarrow x = y$

$\therefore f$ is a one – one function.

It is clear that $f: [-1, 1] \rightarrow$ Range f is onto.

$\therefore f: [-1, 1] \rightarrow$ Range f is one – one and onto and therefore, the inverse of the function $f: [-1, 1] \rightarrow$ Range f exists.

Let g: Range $f \rightarrow [-1, 1]$ be the inverse of f.

Let y be an arbitrary element of range f.

Since $f: [-1, 1] \rightarrow$ Range f is onto, we have

$y = f(x)$ for some $x \in [-1, 1]$

$\Rightarrow y = \frac{x}{x+2}$

$\Rightarrow xy + 2y = x$

$\Rightarrow x(1\text{-}y) = 2y$

$\Rightarrow x = \frac{2y}{1\text{-}y}, \quad y \neq 1$

Now, let us define g: Range $f \rightarrow [-1, 1]$ as

$g(y) = \frac{2y}{1\text{-}y}, \quad y \neq 1$

Now,

$$(gof)(x) = g(f(x)) = g\left(\frac{x}{x+2}\right) = \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} = \frac{2x}{x+2-x} = \frac{2x}{2} = x$$

and

$$(fog)(y) = f(g(y)) = f\left(\frac{2y}{1-y}\right) = \frac{\frac{2y}{1-y}}{\frac{2y}{1-y}+2} = \frac{2y}{2y+2-2y} = \frac{2y}{2} = y$$

$\therefore$ gof $= x = I_{[-1,1]}$ and fog $= y = I_{\text{Range f}}$

$\therefore f^{-1} = g$

$\Rightarrow f^{-1}(y) = \frac{2y}{1-y}, \quad y \neq 1$

Question 7:

Consider f: **R** → **R** given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f.

Answer 7:

f: **R** → **R** is given by, $f(x) = 4x + 3$

For one – one

Let $f(x) = f(y)$

$\Rightarrow 4x + 3 = 4y + 3$

$\Rightarrow 4x = 4y$

$\Rightarrow x = y$

$\therefore f$ is a one – one function.

For onto

For $y \in$ **R**, let $y = 4x + 3$.

$\Rightarrow x = \frac{y-3}{4} \in$ **R**

Therefore, for any $y \in$ **R**, there exists $x = \frac{y-3}{4} \in$ **R**, such that

$\mathrm{f(x) = f\left(\frac{y\text{-}3}{4}\right) = 4\left(\frac{y\text{-}3}{4}\right) + 3 = y.}$

$\therefore$ f is onto.

Thus, f is one – one and onto and therefore, f^{-1} exists.

Let us define g: **R** → **R** by $\mathrm{g(x) = \frac{y\text{-}3}{4}}$

Now,

$$\mathrm{(gof)(x) = g\big(f(x)\big) = g(4x + 3) = \frac{(4x + 3)\text{-}3}{4} = \frac{4x}{4} = x}$$

and

$$\mathrm{(fog)(y) = f\big(g(y)\big) = f\left(\frac{y\text{-}3}{4}\right) = 4\left(\frac{y\text{-}3}{4}\right) + 3 = y\text{-}3 + 3 = y}$$

$\therefore$ $\mathrm{gof = fog = I_R}$

Hence, f is invertible and the inverse of f is given by $\mathrm{f^{-1}(y) = g(y) = \frac{y\text{-}3}{4}}$.

Question 8:

Consider f: $\boldsymbol{R}_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of given f by $f^{-1}(y) = \sqrt{y-4}$, where $\boldsymbol{R}_+$ is the set of all non-negative real numbers.

Answer 8:

f: $\boldsymbol{R}_+ \to [4, \infty)$ is given as $f(x) = x^2 + 4$.

For one – one

Let $f(x) = f(y)$

$\Rightarrow \mathrm{x^2 + 4 = y^2 + 4}$

$\Rightarrow \mathrm{x^2 = y^2}$

$\Rightarrow \mathrm{x = y}$ [$as\ x = y\ \in \boldsymbol{R}_+$]

$\therefore$ f is a one – one function.

For onto

For $y \in [4, \infty)$, let $y = x^2 + 4$

$\Rightarrow x^2 = y - 4 \geq 0$ [as $y \geq 4$]

$\Rightarrow x = \sqrt{y-4} \geq 0$

Therefore, for any $y \in [4, \infty)$, there exists $x = \sqrt{y-4} \in R_+$, such that

$f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y-4+4 = y$

$\therefore f$ is onto.

Thus, f is one – one and onto and therefore, f^{-1} exists.

Let us define g: $[4, \infty) \rightarrow R_+$ by $g(y) = \sqrt{y-4}$

Now,

$$(gof)(x) = g(f(x)) = g(x^2+4) = \sqrt{(x^2+4)-4} = \sqrt{x^2} = x$$

and

$$(fog)(y) = f(g(y)) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y-4+4 = y$$

$\therefore gof = fog = I_R$

Hence, f is invertible and the inverse of f is given by $f^{-1}(y) = g(y) = \sqrt{y-4}$

Question 9:

Consider f: $\boldsymbol{R}_+ \rightarrow [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{(\sqrt{y+6})-1}{3}\right)$

Answer 9:

f: $R_+ \rightarrow [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

Let y be an arbitrary element of $[-5, \infty)$.

Let $y = 9x^2 + 6x - 5$

$\Rightarrow y = (3x+1)^2 - 1 - 5 = (3x+1)^2 - 6$

$\Rightarrow y + 6 = (3x+1)^2$

$\Rightarrow 3x + 1 = \sqrt{y+6}$ [as $y \geq -5 \Rightarrow y + 6 > 0$]

$\Rightarrow x = \frac{(\sqrt{y+6})-1}{3}$

$\therefore$ f is onto, thereby range $f = [-5, \infty)$.

Let us define g: $[-5, \infty) \rightarrow R_+$ as $g(y) = \frac{(\sqrt{y+6})-1}{3}$

Now,

$$(gof)(x) = g(f(x)) = g(9x^2 + 6x\text{-}5) = g((3x+1)^2\text{-}6)$$
$$= \sqrt{(3x+1)^2\text{-}6+6}\text{-}1$$
$$= \frac{3x+1\text{-}1}{3} = \frac{3x}{3} = x$$

and

$$(fog)(y) = f(g(y)) = f\left(\frac{\sqrt{y+6}\text{-}1}{3}\right) = \left[3\left(\frac{\sqrt{y+6}\text{-}1}{3}\right)+1\right]^2\text{-}6$$
$$= \left(\sqrt{y+6}\right)^2\text{-}6 = y+6\text{-}6 = y$$

$\therefore$ $gof = x = I_R$ and $fog = y = I_{\text{Range f}}$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \left(\frac{(\sqrt{y+6})-1}{3}\right)$$

Question 10:

Let f: $X \rightarrow Y$ be an invertible function. Show that f has unique inverse. (Hint: suppose g_1 and g_2 are two inverses of f. Then for all $y \in Y$, $fog_1(y) = I_Y(y) = fog_2(y)$. Use one – one ness of f).

Answer 10:

Let $f: X \to Y$ be an invertible function.

Also, suppose f has two inverses (say g_1 and g_2)

Then, for all $y \in Y$, we have

$fog_1(y) = I_Y(y) = fog_2(y)$

$\Rightarrow f(g_1(y)) = f(g_2(y))$

$\Rightarrow g_1(y) = g_2(y)$ [as f is invertible $\Rightarrow f$ is one – one]

$\Rightarrow g_1 = g_2$ [as g is one – one]

Hence, f has a unique inverse.

Question 11:

Consider $f: \{1, 2, 3\} \to \{a, b, c\}$ given by $f(1) = a$, $f(2) = b$ and $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

Answer 11:

Function $f: \{1, 2, 3\} \to \{a, b, c\}$ is given by $f(1) = a$, $f(2) = b$, and $f(3) = c$

If we define $g: \{a, b, c\} \to \{1, 2, 3\}$ as $g(a) = 1$, $g(b) = 2$, $g(c) = 3$.

We have

$$(fog)(a) = f(g(a)) = f(1) = a$$

$$(fog)(b) = f(g(b)) = f(2) = b$$

$$(fog)(c) = f(g(c)) = f(3) = c$$

and

$$(gof)(1) = g(f(1)) = f(a) = 1$$

$$(gof)(2) = g(f(2)) = f(b) = 2$$

$$(gof)(3) = g(f(3)) = f(c) = 3$$

$\therefore$ gof $= I_X$ and fog $= I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.
Thus, the inverse of f exists and $f^{-1} = g$.

$\therefore f^{-1}:\{a, b, c\} \to \{1, 2, 3\}$ is given by $f^{-1}(a) = 1, f^{-1}(b) = 2, f^{-1}(c) = 3$

Let us now find the inverse of f^{-1} i.e., find the inverse of g.

If we define $h: \{1, 2, 3\} \to \{a, b, c\}$ as $h(1) = a, h(2) = b, h(3) = c$

We have

$$(\mathrm{goh})(1) = \mathrm{g}\big(\mathrm{h}(1)\big) = \mathrm{g}(\mathrm{a}) = 1$$

$$(\mathrm{goh})(2) = \mathrm{g}\big(\mathrm{h}(2)\big) = \mathrm{g}(\mathrm{b}) = 2$$

$$(\mathrm{goh})(3) = \mathrm{g}\big(\mathrm{h}(3)\big) = \mathrm{g}(\mathrm{c}) = 3$$

and

$$(\mathrm{hog})(\mathrm{a}) = \mathrm{h}\big(\mathrm{g}(\mathrm{a})\big) = \mathrm{h}(1) = \mathrm{a}$$

$$(\mathrm{hog})(\mathrm{b}) = \mathrm{h}\big(\mathrm{g}(\mathrm{b})\big) = \mathrm{h}(2) = \mathrm{b}$$

$$(\mathrm{hog})(\mathrm{c}) = \mathrm{h}\big(\mathrm{g}(\mathrm{c})\big) = \mathrm{h}(3) = \mathrm{c}$$

$\therefore$ goh $= I_X$ and hog $= I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Thus, the inverse of g exists and $g^{-1} = h \Rightarrow (f^{-1})^{-1} = h$.

It can be noted that $h = f$.

Hence, $(f^{-1})^{-1} = f$.

Question 12:

Let $f: X \to Y$ be an invertible function. Show that the inverse of f^{-1} is f,

i.e., $(f^{-1})^{-1} = f$.

Answer 12:

Let $f: X \rightarrow Y$ be an invertible function.

Then, there exists a function g: $Y \rightarrow X$ such that $gof = I_X$ and $fog = I_Y$.

Here, $f^{-1} = g$.

Now, $gof = I_X$ and $fog = I_Y$

$\Rightarrow f^{-1}of = I_X$ and $fof^{-1} = I_Y$

Hence, $f^{-1}: Y \rightarrow X$ is invertible and f is the inverse of f^{-1} i.e., $(f^{-1})^{-1} = f$.

Question 13:

If f: **R** $\rightarrow$ **R** be given by $f(x) = (3 - x^3)^{\frac{1}{3}}$, then $fof(x)$ is

(A) $\frac{1}{x^3}$ (B) x^3 (C) x (D) $(3 - x^3)$

Answer 13:

f: **R** $\rightarrow$ **R** be given as $f(x) = (3\text{-}x^3)^{\frac{1}{3}}$

$$\therefore fof(x) = f\big(f(x)\big) = f\left((3\text{-}x^3)^{\frac{1}{3}}\right) = \left[3\text{-}\left((3\text{-}x^3)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$$

$$= [3\text{-}(3\text{-}x^3)]^{\frac{1}{3}} = (x^3)^{\frac{1}{3}} = x$$

$\therefore fof(x) = x$

The correct answer is C.

Question 14: Let f: R-$\left\{-\frac{4}{3}\right\}$ → R be a function as f(x) $= \frac{4x}{3x+4}$. The inverse of *f* is map *g*: Range f → R-$\left\{-\frac{4}{3}\right\}$ given by

(A) $g(y) = \frac{3y}{3-4y}$

(B) $g(y) = \frac{4y}{4-3y}$

(C) $g(y) = \frac{4y}{3-4y}$

(D) $g(y) = \frac{3y}{4-3y}$

Answer 14:

It is given that $f: \boldsymbol{R} - \left\{-\frac{4}{3}\right\} \to \boldsymbol{R}$ be a function as $f(x) = \frac{4x}{3x+4}$

Let *y* be an arbitrary element of Range *f*.

Then, there exists $x \in$ R-$\left\{-\frac{4}{3}\right\}$ such that y = f(x)

$$\Rightarrow y = \frac{4x}{3x+4}$$

$$\Rightarrow 3xy + 4y = 4x$$

$$\Rightarrow x(4\text{-}3y) = 4y$$

$$\Rightarrow x = \frac{4y}{4\text{-}3y}$$

Let us define g: Range f → R-$\left\{-\frac{4}{3}\right\}$ as $g(y) = \frac{4y}{4-3y}$

Now,

$$gof(x) = g(f(x)) = g\left(\frac{4x}{3x+4}\right) = \frac{4\left(\frac{4x}{3x+4}\right)}{4 - 3\left(\frac{4x}{3x+4}\right)}$$

$$= \frac{16x}{12x + 16 - 12x} = \frac{16x}{16} = x$$

and

$$fog(y) = f(g(y)) = f\left(\frac{4y}{4-3y}\right) = \frac{4\left(\frac{4y}{4-3y}\right)}{3\left(\frac{4y}{4-3y}\right)+4}$$

$$= \frac{16y}{12y+16-12y} = \frac{16y}{16} = y$$

$\therefore$ gof $= I_{R-\left\{-\frac{4}{3}\right\}}$ and fog $= I_{\text{Range f}}$

Thus, g is the inverse of *f* i.e., $f^{-1} = g$.

Hence, the inverse of *f* is the map *g*: Range f → R-$\left\{-\frac{4}{3}\right\}$, which is given by g(y) = $\frac{4y}{4-3y}$.

The correct answer is B.