Exercise 11.2

Question 1:

Show that the three lines with direction cosines

$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

Answer

Two lines with direction cosines, l_1, m_1, n_1 and l_2, m_2, n_2, are perpendicular to each other, if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

(i) For the lines with direction cosines, $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ and $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \frac{12}{13}\times\frac{4}{13} + \left(\frac{-3}{13}\right)\times\frac{12}{13} + \left(\frac{-4}{13}\right)\times\frac{3}{13} \\ &= \frac{48}{169} - \frac{36}{169} - \frac{12}{169} \\ &= 0 \end{aligned}$$

,

Therefore, the lines are perpendicular.

(ii) For the lines with direction cosines, $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \frac{4}{13}\times\frac{3}{13} + \frac{12}{13}\times\left(\frac{-4}{13}\right) + \frac{3}{13}\times\frac{12}{13} \\ &= \frac{12}{169} - \frac{48}{169} + \frac{36}{169} \\ &= 0 \end{aligned}$$

Therefore, the lines are perpendicular.

(iii) For the lines with direction cosines, $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ and $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$, we obtain

$$l_1l_2 + m_1m_2 + n_1n_2 = \left(\frac{3}{13}\right)\times\left(\frac{12}{13}\right)+\left(\frac{-4}{13}\right)\times\left(\frac{-3}{13}\right)+\left(\frac{12}{13}\right)\times\left(\frac{-4}{13}\right)$$
$$= \frac{36}{169}+\frac{12}{169}-\frac{48}{169}$$
$$= 0$$

Therefore, the lines are perpendicular.

Thus, all the lines are mutually perpendicular.

Question 2:

Show that the line through the points (1, −1, 2) (3, 4, −2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

Answer

Let AB be the line joining the points, (1, −1, 2) and (3, 4, − 2), and CD be the line joining the points, (0, 3, 2) and (3, 5, 6).

The direction ratios, a_1, b_1, c_1, of AB are (3 − 1), (4 − (−1)), and (−2 − 2) i.e., 2, 5, and −4.

The direction ratios, a_2, b_2, c_2, of CD are (3 − 0), (5 − 3), and (6 −2) i.e., 3, 2, and 4. AB and CD will be perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ $a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 3 + 5 \times 2 + (-4) \times 4$

$= 6 + 10 - 16$

$= 0$

Therefore, AB and CD are perpendicular to each other.

Question 3:

Show that the line through the points (4, 7, 8) (2, 3, 4) is parallel to the line through the points (−1, −2, 1), (1, 2, 5).

Answer

Let AB be the line through the points, (4, 7, 8) and (2, 3, 4), and CD be the line through the points, (−1, −2, 1) and (1, 2, 5).

The directions ratios, a_1, b_1, c_1, of AB are (2 − 4), (3 − 7), and (4 − 8) i.e., −2, −4, and −4.

The direction ratios, a_2, b_2, c_2, of CD are (1 − (−1)), (2 − (−2)), and (5 − 1) i.e., 2, 4, and 4.

AB will be parallel to CD, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$$\frac{a_1}{a_2} = \frac{-2}{2} = -1$$

$$\frac{b_1}{b_2} = \frac{-4}{4} = -1$$

$$\frac{c_1}{c_2} = \frac{-4}{4} = -1$$

$$\therefore \ \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, AB is parallel to CD.

Question 4:

Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

Answer

It is given that the line passes through the point A (1, 2, 3). Therefore, the position vector through A is $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$$

It is known that the line which passes through point A and parallel to $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a constant.

$$\Rightarrow \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(3\hat{i} + 2\hat{j} - 2\hat{k}\right)$$

This is the required equation of the line.

Question 5:

Find the equation of the line in vector and in Cartesian form that passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$.

Answer

It is given that the line passes through the point with position vector

$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$ ...(1)

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ ...(2)

It is known that a line through a point with position vector $\vec{a}$ and parallel to $\vec{b}$ is given by

the equation, $\vec{r} = \vec{a} + \lambda\vec{b}$

$\Rightarrow \vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$

This is the required equation of the line in vector form.

$\vec{r} = x\hat{i} - y\hat{j} + z\hat{k}$

$\Rightarrow x\hat{i} - y\hat{j} + z\hat{k} = (\lambda + 2)\hat{i} + (2\lambda - 1)\hat{j} + (-\lambda + 4)\hat{k}$

Eliminating λ, we obtain the Cartesian form equation as

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

This is the required equation of the given line in Cartesian form.

Question 6:

Find the Cartesian equation of the line which passes through the point

(−2, 4, −5) and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Answer

It is given that the line passes through the point (−2, 4, −5) and is parallel to

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$$

The direction ratios of the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$, , are 3, 5, and 6.

The required line is parallel to $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Therefore, its direction ratios are 3k, 5k, and 6k, where $k \neq 0$

It is known that the equation of the line through the point (x_1, y_1, z_1) and with direction ratios, a, b, c, is given by $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

Therefore the equation of the required line is

$$\frac{x+2}{3k}=\frac{y-4}{5k}=\frac{z+5}{6k}$$

$$\Rightarrow \frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}=k$$

Question 7:

The Cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Write its vector form.

Answer

The Cartesian equation of the line is

$$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2} \qquad \ldots(1)$$

The given line passes through the point (5, −4, 6). The position vector of this point is $\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$

Also, the direction ratios of the given line are 3, 7, and 2.

This means that the line is in the direction of vector, $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

It is known that the line through position vector $\vec{a}$ and in the direction of the vector $\vec{b}$ is given by the equation, $\vec{r}=\vec{a}+\lambda\vec{b}, \lambda \in R$

$$\Rightarrow \vec{r}=\left(5\hat{i}-4\hat{j}+6\hat{k}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$$

This is the required equation of the given line in vector form.

Question 8:

Find the vector and the Cartesian equations of the lines that pass through the origin and (5, −2, 3).

Answer

The required line passes through the origin. Therefore, its position vector is given by,

$\vec{a} = \vec{0}$ ... (1)

The direction ratios of the line through origin and (5, −2, 3) are

(5 − 0) = 5, (−2 − 0) = −2, (3 − 0) = 3

The line is parallel to the vector given by the equation, $\vec{b} = 5\hat{i} - 2\hat{j} + 3\hat{k}$

The equation of the line in vector form through a point with position vector $\vec{a}$ and parallel

to $\vec{b}$ is, $\vec{r} = \vec{a} + \lambda\vec{b},\ \lambda \in R$

$$\Rightarrow \vec{r} = \vec{0} + \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$$

$$\Rightarrow \vec{r} = \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$$

The equation of the line through the point (x_1, y_1, z_1) and direction ratios a, b, c is given

by, $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

Therefore, the equation of the required line in the Cartesian form is

$$\frac{x - 0}{5} = \frac{y - 0}{-2} = \frac{z - 0}{3}$$

$$\Rightarrow \frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$$

Question 9:

Find the vector and the Cartesian equations of the line that passes through the points

(3, −2, −5), (3, −2, 6).

Answer

Let the line passing through the points, P (3, −2, −5) and Q (3, −2, 6), be PQ.

Since PQ passes through P (3, −2, −5), its position vector is given by,

$$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$$

The direction ratios of PQ are given by,

(3 − 3) = 0, (−2 + 2) = 0, (6 + 5) = 11

The equation of the vector in the direction of PQ is

$$\vec{b} = 0.\hat{i} - 0.\hat{j} + 11\hat{k} = 11\hat{k}$$

The equation of PQ in vector form is given by, $\vec{r}=\vec{a}+\lambda\vec{b},\ \lambda\in R$

$$\Rightarrow \vec{r}=\left(3\hat{i}-2\hat{j}-5\hat{k}\right)+11\lambda\hat{k}$$

The equation of PQ in Cartesian form is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

$$\frac{x-3}{0}=\frac{y+2}{0}=\frac{z+5}{11}$$

Question 10:

Find the angle between the following pairs of lines:

(i) $\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda\left(3\hat{i}-2\hat{j}+6\hat{k}\right)$ and $\vec{r}=7\hat{i}-6\hat{k}+\mu\left(\hat{i}+2\hat{j}+2\hat{k}\right)$

(ii) $\vec{r}=3\hat{i}+\hat{j}-2\hat{k}+\lambda\left(\hat{i}-\hat{j}-2\hat{k}\right)$ and $\vec{r}=2\hat{i}-\hat{j}-56\hat{k}+\mu\left(3\hat{i}-5\hat{j}-4\hat{k}\right)$

Answer

(i) Let Q be the angle between the given lines.

The angle between the given pairs of lines is given by, $\cos Q=\left|\frac{\vec{b}_1\cdot\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

The given lines are parallel to the vectors, , $\vec{b}_1=3\hat{i}+2\hat{j}+6\hat{k}$ and $\vec{b}_2=\hat{i}+2\hat{j}+2\hat{k}$ respectively.

$\therefore |\vec{b}_1| = \sqrt{3^2 + 2^2 + 6^2} = 7$

$|\vec{b}_2| = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$

$\vec{b}_1 \cdot \vec{b}_2 = (3\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k})$

$= 3\times 1 + 2\times 2 + 6\times 2$

$= 3 + 4 + 12$

$= 19$

$\Rightarrow \cos Q = \dfrac{19}{7\times 3}$

$\Rightarrow Q = \cos^{-1}\left(\dfrac{19}{21}\right)$

(ii) The given lines are parallel to the vectors, $\vec{b}_1 = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}$, respectively.

$\therefore |\vec{b}_1| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$

$|\vec{b}_2| = \sqrt{(3)^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$

$\vec{b}_1 \cdot \vec{b}_2 = (\hat{i} - \hat{j} - 2\hat{k}) \cdot (3\hat{i} - 5\hat{j} - 4\hat{k})$

$= 1\cdot 3 - 1(-5) - 2(-4)$

$= 3 + 5 + 8$

$= 16$

$\cos Q = \left|\dfrac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

$\Rightarrow \cos Q = \dfrac{16}{\sqrt{6}\cdot 5\sqrt{2}} = \dfrac{16}{\sqrt{2}\cdot\sqrt{3}\cdot 5\sqrt{2}} = \dfrac{16}{10\sqrt{3}}$

$\Rightarrow \cos Q = \dfrac{8}{5\sqrt{3}}$

$\Rightarrow Q = \cos^{-1}\left(\dfrac{8}{5\sqrt{3}}\right)$

Question 11:

Find the angle between the following pairs of lines:

(i) $\frac{x-2}{2}=\frac{y-1}{5}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$

(ii) $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x-5}{4}=\frac{y-2}{1}=\frac{z-3}{8}$

Answer

i. Let $\vec{b}_1$ and $\vec{b}_2$ be the vectors parallel to the pair of lines,

$$\frac{x-2}{2}=\frac{y-1}{5}=\frac{z+3}{-3} \text{ and } \frac{x+2}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$$

$\therefore \vec{b}_1=2\hat{i}+5\hat{j}-3\hat{k}$ and $\vec{b}_2=-\hat{i}+8\hat{j}+4\hat{k}$

$$\left|\vec{b}_1\right|=\sqrt{(2)^2+(5)^2+(-3)^2}=\sqrt{38}$$

$$\left|\vec{b}_2\right|=\sqrt{(-1)^2+(8)^2+(4)^2}=\sqrt{81}=9$$

$$\begin{aligned}\vec{b}_1\cdot\vec{b}_2&=\left(2\hat{i}+5\hat{j}-3\hat{k}\right)\cdot\left(-\hat{i}+8\hat{j}+4\hat{k}\right)\\&=2(-1)+5\times 8+(-3)\cdot 4\\&=-2+40-12\\&=26\end{aligned}$$

The angle, Q, between the given pair of lines is given by the relation,

$$\cos Q=\left|\frac{\vec{b}_1\cdot\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$$

$$\Rightarrow \cos Q=\frac{26}{9\sqrt{38}}$$

$$\Rightarrow Q=\cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

(ii) Let $\vec{b}_1, \vec{b}_2$ be the vectors parallel to the given pair of lines,

$\frac{x-5}{4} = \frac{y-5}{1} = \frac{z-3}{8}$ and $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ respectively.

$$\vec{b}_1 = 2\hat{i} + 2\hat{j} + \hat{k}$$
$$\vec{b}_2 = 4\hat{i} + \hat{j} + 8\hat{k}$$
$$\therefore \left|\vec{b}_1\right| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{9} = 3$$
$$\left|\vec{b}_2\right| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$$
$$\begin{aligned}\vec{b}_1 \cdot \vec{b}_2 &= \left(2\hat{i} + 2\hat{j} + \hat{k}\right) \cdot \left(4\hat{i} + \hat{j} + 8\hat{k}\right) \\ &= 2\times4 + 2\times1 + 1\times8 \\ &= 8 + 2 + 8 \\ &= 18\end{aligned}$$

If Q is the angle between the given pair of lines, then $\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

$$\Rightarrow \cos Q = \frac{18}{3\times9} = \frac{2}{3}$$
$$\Rightarrow Q = \cos^{-1}\left(\frac{2}{3}\right)$$

Question 12:

Find the values of p so the line $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.

Answer

The given equations can be written in the standard form as

$$\frac{x-1}{-3}=\frac{y-2}{\frac{2p}{7}}=\frac{z-3}{2} \quad \text{and} \quad \frac{x-1}{\frac{-3p}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}$$

The direction ratios of the lines are $-3, \frac{2p}{7}$, 2 and $\frac{-3p}{7}, 1, -5$ respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular to each other, if $a_1a_2 + b_1 b_2 + c_1c_2 = 0$

$$\therefore (-3)\cdot\left(\frac{-3p}{7}\right)+\left(\frac{2p}{7}\right)\cdot(1)+2\cdot(-5)=0$$

$$\Rightarrow \frac{9p}{7}+\frac{2p}{7}=10$$

$$\Rightarrow 11p=70$$

$$\Rightarrow p=\frac{70}{11}$$

Thus, the value of p is $\frac{70}{11}$.

Question 13:

Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are perpendicular to each other.

Answer

The equations of the given lines are $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

The direction ratios of the given lines are 7, −5, 1 and 1, 2, 3 respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular to each other, if $a_1a_2 + b_1 b_2 + c_1c_2 = 0$

$\therefore 7 \times 1 + (-5) \times 2 + 1 \times 3$

$= 7 - 10 + 3$

$= 0$

Therefore, the given lines are perpendicular to each other.

Question 14:

Find the shortest distance between the lines

$\vec{r}=\left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$ and

$\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$

Answer

The equations of the given lines are

$\vec{r}=\left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$

$\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$

It is known that the shortest distance between the lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$, is given by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_2\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right| \qquad ...(1)$$

Comparing the given equations, we obtain

$\vec{a}_1=\hat{i}+2\hat{j}+\hat{k}$

$\vec{b}_1=\hat{i}-\hat{j}+\hat{k}$

$\vec{a}_2=2\hat{i}-\hat{j}-\hat{k}$

$\vec{b}_2=2\hat{i}+\hat{j}+2\hat{k}$

$\vec{a}_2-\vec{a}_1=\left(2\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}+2\hat{j}+\hat{k}\right)=\hat{i}-3\hat{j}-2\hat{k}$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 2 & 1 & 2\end{vmatrix}$$

$\vec{b}_1\times\vec{b}_2=(-2-1)\hat{i}-(2-2)\hat{j}+(1+2)\hat{k}=-3\hat{i}+3\hat{k}$

$\Rightarrow\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(-3)^2+(3)^2}=\sqrt{9+9}=\sqrt{18}=3\sqrt{2}$

Substituting all the values in equation (1), we obtain

$$d=\left|\frac{\left(-3\hat{i}+3\hat{k}\right)\cdot\left(\hat{i}-3\hat{j}-2\hat{k}\right)}{3\sqrt{2}}\right|$$

$$\Rightarrow d=\left|\frac{-3.1+3(-2)}{3\sqrt{2}}\right|$$

$$\Rightarrow d=\left|\frac{-9}{3\sqrt{2}}\right|$$

$$\Rightarrow d=\frac{3}{\sqrt{2}}=\frac{3\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the two lines is $\frac{3\sqrt{2}}{2}$ units.

Question 15:

Find the shortest distance between the lines

$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

Answer

The given lines are

$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

It is known that the shortest distance between the two lines,

$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$, is given by,

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}} \qquad \ldots(1)$$

Comparing the given equations, we obtain

$$\begin{aligned} &x_1=-1,\ y_1=-1,\ z_1=-1 \\ &a_1=7,\ b_1=-6,\ c_1=1 \\ &x_2=3,\ y_2=5,\ z_2=7 \\ &a_2=1,\ b_2=-2,\ c_2=1 \end{aligned}$$

$$\text{Then,}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=\begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$\begin{aligned} &=4(-6+2)-6(7-1)+8(-14+6) \\ &=-16-36-64 \\ &=-116 \end{aligned}$$

$$\begin{aligned} \Rightarrow \sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2} &=\sqrt{(-6+2)^2+(1+7)^2+(-14+6)^2} \\ &=\sqrt{16+36+64} \\ &=\sqrt{116} \\ &=2\sqrt{29} \end{aligned}$$

Substituting all the values in equation (1), we obtain

$$d=\frac{-116}{2\sqrt{29}}=\frac{-58}{\sqrt{29}}=\frac{-2\times 29}{\sqrt{29}}=-2\sqrt{29}$$

Since distance is always non-negative, the distance between the given lines is $2\sqrt{29}$ units.

Question 16:

Find the shortest distance between the lines whose vector equations are

$\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(\hat{i}-3\hat{j}+2\hat{k}\right)$

and $\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu\left(2\hat{i}+3\hat{j}+\hat{k}\right)$

Answer

$$\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda\left(\hat{i}-3\hat{j}+2\hat{k}\right) \quad \text{and} \quad \vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu\left(2\hat{i}+3\hat{j}+\hat{k}\right)$$

It is known that the shortest distance between the lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$

The given lines are

given by,

$$d = \left| \frac{\left(\vec{b}_1 \times \vec{b}_2\right) \cdot \left(\vec{a}_2 - \vec{a}_2\right)}{\left|\vec{b}_1 \times \vec{b}_2\right|} \right| \qquad ...(1)$$

Comparing the given equations with $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$

$\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$

$\vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$

$\vec{a}_2 - \vec{a}_1 = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = 3\hat{i} + 3\hat{j} + 3\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-3-6)\hat{i} - (1-4)\hat{j} + (3+6)\hat{k} = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\Rightarrow \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$$

$$\begin{aligned} \left(\vec{b}_1 \times \vec{b}_2\right) \cdot \left(\vec{a}_2 - \vec{a}_1\right) &= \left(-9\hat{i} + 3\hat{j} + 9\hat{k}\right) \cdot \left(3\hat{i} + 3\hat{j} + 3\hat{k}\right) \\ &= -9 \times 3 + 3 \times 3 + 9 \times 3 \\ &= 9 \end{aligned}$$

Substituting all the values in equation (1), we obtain

$$d = \left| \frac{9}{3\sqrt{19}} \right| = \frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the two given lines is $\frac{3}{\sqrt{19}}$ units.

Question 17:

Find the shortest distance between the lines whose vector equations are

$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ and

$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$

Answer

The given lines are

$$\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$$

$$\Rightarrow \vec{r}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)+t\left(-\hat{i}+\hat{j}-2\hat{k}\right) \qquad ...(1)$$

$$\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$$

$$\Rightarrow \vec{r}=\left(\hat{i}-\hat{j}+\hat{k}\right)+s\left(\hat{i}+2\hat{j}-2\hat{k}\right) \qquad ...(2)$$

It is known that the shortest distance between the lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$, is given by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_2\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right| \qquad ...(3)$$

For the given equations,

$$\vec{a}_1=\hat{i}-2\hat{j}+3\hat{k}$$

$$\vec{b}_1=-\hat{i}+\hat{j}-2\hat{k}$$

$$\vec{a}_2=\hat{i}-\hat{j}-\hat{k}$$

$$\vec{b}_2=\hat{i}+2\hat{j}-2\hat{k}$$

$$\vec{a}_2-\vec{a}_1=\left(\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}-2\hat{j}+3\hat{k}\right)=\hat{j}-4\hat{k}$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & -2\\ 1 & 2 & -2\end{vmatrix}=(-2+4)\hat{i}-(2+2)\hat{j}+(-2-1)\hat{k}=2\hat{i}-4\hat{j}-3\hat{k}$$

$$\Rightarrow\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(2)^2+(-4)^2+(-3)^2}=\sqrt{4+16+9}=\sqrt{29}$$

$$\therefore\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_1\right)=\left(2\hat{i}-4\hat{j}-3\hat{k}\right)\cdot\left(\hat{j}-4\hat{k}\right)=-4+12=8$$

Substituting all the values in equation (3), we obtain

$$d=\left|\frac{8}{\sqrt{29}}\right|=\frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.