Question 1:

Exercise 13.5

A die is thrown 6 times. If 'getting an odd number' is a success, what is the probability of

(i) 5 successes? (ii) at least 5 successes?

(iii) at most 5 successes?

Answer 1:

The repeated tosses of a die are Bernoulli trials. Let X denote the number of successes of getting odd numbers in an experiment of 6 trials.

Probability of getting an odd number in a single throw of a die is, $p = \frac{3}{6} = \frac{1}{2}$

$\therefore q = 1 - p = \frac{1}{2}$

X has a binomial distribution.

Therefore, P $(X = x) = {}^{n}C_{n-x} q^{n-x} p^{x}$, where $n = 0, 1, 2 \ldots n$

$$= {}^{6}C_{x} \left(\frac{1}{2}\right)^{6-x} \cdot \left(\frac{1}{2}\right)^{x}$$
$$= {}^{6}C_{x} \left(\frac{1}{2}\right)^{6}$$

(i) P (5 successes) = P (X = 5)

$$= {}^{6}C_{5} \left(\frac{1}{2}\right)^{6}$$
$$= 6 \cdot \frac{1}{64}$$
$$= \frac{3}{32}$$

(ii) P(at least 5 successes) = P(X ≥ 5)

$$= P(X = 5) + P(X = 6)$$
$$= {}^{6}C_{5} \left(\frac{1}{2}\right)^{6} + {}^{6}C_{6} \left(\frac{1}{2}\right)^{6}$$
$$= 6 \cdot \frac{1}{64} + 1 \cdot \frac{1}{64}$$
$$= \frac{7}{64}$$

(iii) P (at most 5 successes) = P(X ≤ 5)

$$= 1 - P(X > 5)$$
$$= 1 - P(X = 6)$$
$$= 1 - {}^{6}C_{6} \left(\frac{1}{2}\right)^{6}$$
$$= 1 - \frac{1}{64}$$
$$= \frac{63}{64}$$

Question 2:

A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of two successes.

Answer 2:

The repeated tosses of a pair of dice are Bernoulli trials. Let X denote the number of times of getting doublets in an experiment of throwing two dice simultaneously four times.

Probability of getting doublets in a single throw of the pair of dice is

$$p=\frac{6}{36}=\frac{1}{6}$$

$$\therefore q=1-p=1-\frac{1}{6}=\frac{5}{6}$$

Clearly, X has the binomial distribution with $n=4$, $p=\frac{1}{6}$, and $q=\frac{5}{6}$

$$\therefore P(X=x)={}^{n}C_{x}q^{n-x}p^{x}, \text{ where } x=0, 1, 2, 3 \ldots n$$

$$={}^{4}C_{x}\left(\frac{5}{6}\right)^{4-x}\cdot\left(\frac{1}{6}\right)^{x}$$

$$={}^{4}C_{x}\cdot\frac{5^{4-x}}{6^{4}}$$

$\therefore$ P (2 successes) = P (X = 2)

$$={}^{4}C_{2}\cdot\frac{5^{4-2}}{6^{4}}$$

$$=6\cdot\frac{25}{1296}$$

$$=\frac{25}{216}$$

Question 3:

There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Answer 3:

Let X denote the number of defective items in a sample of 10 items drawn successively. Since the drawing is done with replacement, the trials are Bernoulli trials.

$$\Rightarrow p=\frac{5}{100}=\frac{1}{20}$$

$$\therefore q=1-\frac{1}{20}=\frac{19}{20}$$

X has a binomial distribution with $n=10$ and $p=\frac{1}{20}$

$$P(X=x)={}^{n}C_{x}q^{n-x}p^{x}, \text{ where } x=0, 1, 2 \ldots n$$

$$={}^{10}C_{x}\left(\frac{19}{20}\right)^{10-x}\cdot\left(\frac{1}{20}\right)^{x}$$

P (not more than 1 defective item) = P (X ≤ 1)

$$= P(X=0)+P(X=1)$$
$$= {}^{10}C_0\left(\frac{19}{20}\right)^{10}\cdot\left(\frac{1}{20}\right)^{0} + {}^{10}C_1\left(\frac{19}{20}\right)^{9}\cdot\left(\frac{1}{20}\right)^{1}$$
$$= \left(\frac{19}{20}\right)^{10} + 10\left(\frac{19}{20}\right)^{9}\cdot\left(\frac{1}{20}\right)$$
$$= \left(\frac{19}{20}\right)^{9}\cdot\left[\frac{19}{20}+\frac{10}{20}\right]$$
$$= \left(\frac{19}{20}\right)^{9}\cdot\left(\frac{29}{20}\right)$$
$$= \left(\frac{29}{20}\right)\cdot\left(\frac{19}{20}\right)^{9}$$

Question 4:

Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(i) all the five cards are spades?

(ii) only 3 cards are spades?

(iii) none is a spade?

Answer 4:

Let X represent the number of spade cards among the five cards drawn. Since the drawing of card is with replacement, the trials are Bernoulli trials.

In a well shuffled deck of 52 cards, there are 13 spade cards.

$$\Rightarrow p = \frac{13}{52} = \frac{1}{4}$$
$$\therefore q = 1-\frac{1}{4} = \frac{3}{4}$$

X has a binomial distribution with $n = 5$ and $p = \frac{1}{4}$

$$P(X=x) = {}^{n}C_x q^{n-x} p^{x}, \text{ where } x = 0, 1, \ldots n$$
$$= {}^{5}C_x\left(\frac{3}{4}\right)^{5-x}\left(\frac{1}{4}\right)^{x}$$

(i) P (all five cards are spades) = P(X = 5)

$$= {}^{5}C_5\left(\frac{3}{4}\right)^{0}\cdot\left(\frac{1}{4}\right)^{5}$$
$$= 1\cdot\frac{1}{1024}$$
$$= \frac{1}{1024}$$

(ii) P (only 3 cards are spades) $= P(X=3)$

$$= {}^5C_3 \cdot \left(\frac{3}{4}\right)^2 \cdot \left(\frac{1}{4}\right)^3$$
$$= 10 \cdot \frac{9}{16} \cdot \frac{1}{64}$$
$$= \frac{45}{512}$$

(iii) P (none is a spade) $= P(X=0)$

$$= {}^5C_0 \cdot \left(\frac{3}{4}\right)^5 \cdot \left(\frac{1}{4}\right)^0$$
$$= 1 \cdot \frac{243}{1024}$$
$$= \frac{243}{1024}$$

Question 5:

The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. What is the probability that out of 5 such bulbs

(i) none

(ii) not more than one

(iii) more than one

(iv) at least one

will fuse after 150 days of use.

Answer 5:

Let X represent the number of bulbs that will fuse after 150 days of use in an experiment of 5 trials. The trials are Bernoulli trials.

It is given that, $p = 0.05$

$\therefore q = 1 - p = 1 - 0.05 = 0.95$

X has a binomial distribution with $n = 5$ and $p = 0.05$

$$\therefore P(X = x) = {}^nC_x q^{n-x} p^x, \text{ where } x = 1, 2, \ldots n$$
$$= {}^5C_x (0.95)^{5-x} \cdot (0.05)^x$$

(i) P (none) $= P(X=0)$

$$= {}^5C_0 (0.95)^5 \cdot (0.05)^0$$
$$= 1 \times (0.95)^5$$
$$= (0.95)^5$$

(ii) P (not more than one) = $P(X \le 1)$

$$= P(X=0)+P(X=1)$$
$$= {}^5C_0(0.95)^5 \times (0.05)^0 + {}^5C_1(0.95)^4 \times (0.05)^1$$
$$= 1\times(0.95)^5 + 5\times(0.95)^4 \times (0.05)$$
$$= (0.95)^5 + (0.25)(0.95)^4$$
$$= (0.95)^4[0.95+0.25]$$
$$= (0.95)^4 \times 1.2$$

(iii) P (more than 1) = $P(X > 1)$

$$= 1-P(X \le 1)$$
$$= 1-P(\text{not more than } 1)$$
$$= 1-(0.95)^4 \times 1.2$$

(iv) P (at least one) = $P(X \ge 1)$

$$= 1-P(X<1)$$
$$= 1-P(X=0)$$
$$= 1-{}^5C_0(0.95)^5 \times (0.05)^0$$
$$= 1-1\times(0.95)^5$$
$$= 1-(0.95)^5$$

Question 6:

A bag consists of 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?

Answer 6:

Let X denote the number of balls marked with the digit 0 among the 4 balls drawn.

Since the balls are drawn with replacement, the trials are Bernoulli trials.

X has a binomial distribution with $n = 4$ and $p = \frac{1}{10}$

$$\therefore q = 1-p = 1-\frac{1}{10} = \frac{9}{10}$$

$$\therefore P(X=x) = {}^nC_x q^{n-x} \cdot p^x, x = 1, 2, \ldots n$$
$$= {}^4C_x\left(\frac{9}{10}\right)^{4-x} \cdot \left(\frac{1}{10}\right)^x$$

P (none marked with 0) = P (X = 0)

$$= {}^4C_0\left(\frac{9}{10}\right)^4 \cdot \left(\frac{1}{10}\right)^0$$
$$= 1\cdot\left(\frac{9}{10}\right)^4$$
$$= \left(\frac{9}{10}\right)^4$$

Question 7:

In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers 'true'; if it falls tails, he answers 'false'. Find the probability that he answers at least 12 questions correctly.

Answer 7:

Let X represent the number of correctly answered questions out of 20 questions.

The repeated tosses of a coin are Bernoulli trails. Since "head" on a coin represents the true answer and "tail" represents the false answer, the correctly answered questions are Bernoulli trials.

$$\therefore p = \frac{1}{2}$$

$$\therefore q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$$

X has a binomial distribution with $n = 20$ and $p = \frac{1}{2}$

$$\therefore P(X = x) = {}^{n}C_x q^{n-x} p^x, \text{ where } x = 0, 1, 2, \dots n$$
$$= {}^{20}C_x \left(\frac{1}{2}\right)^{20-x} \cdot \left(\frac{1}{2}\right)^x$$
$$= {}^{20}C_x \left(\frac{1}{2}\right)^{20}$$

P (at least 12 questions answered correctly) $= P(X \geq 12)$

$$= P(X = 12) + P(X = 13) + \dots + P(X = 20)$$
$$= {}^{20}C_{12}\left(\frac{1}{2}\right)^{20} + {}^{20}C_{13}\left(\frac{1}{2}\right)^{20} + \dots + {}^{20}C_{20}\left(\frac{1}{2}\right)^{20}$$
$$= \left(\frac{1}{2}\right)^{20} \cdot \left[{}^{20}C_{12} + {}^{20}C_{13} + \dots + {}^{20}C_{20}\right]$$

Question 8:

Suppose X has a binomial distribution $B\left(6, \frac{1}{2}\right)$. Show that $X = 3$ is the most likely outcome.

(Hint: $P(X = 3)$ is the maximum among all $P(x_i)$, $x_i = 0, 1, 2, 3, 4, 5, 6$)

Answer 8:

X is the random variable whose binomial distribution is $B\left(6, \frac{1}{2}\right)$.

Therefore, $n = 6$ and $p = \frac{1}{2}$

$$\therefore q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$$

Then, $P(X = x) = {}^{n}C_x q^{n-x} p^x$

$$= {}^{6}C_x \left(\frac{1}{2}\right)^{6-x} \cdot \left(\frac{1}{2}\right)^x$$

$$= {}^6C_x \left(\frac{1}{2}\right)^6$$

It can be seen that $P(X = x)$ will be maximum, if ${}^6C_x$ will be maximum.

Then, ${}^6C_0 = {}^6C_6 = \frac{6!}{0!\cdot 6!} = 1$

${}^6C_1 = {}^6C_5 = \frac{6!}{1!\cdot 5!} = 6$

${}^6C_2 = {}^6C_4 = \frac{6!}{2!\cdot 4!} = 15$

${}^6C_3 = \frac{6!}{3!\cdot 3!} = 20$

The value of ${}^6C_3$ is maximum. Therefore, for $x = 3$, $P(X = x)$ is maximum.

Thus, $X = 3$ is the most likely outcome.

Question 9:

On a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

Answer 9:

The repeated guessing of correct answers from multiple choice questions are Bernoulli trials. Let X represent the number of correct answers by guessing in the set of 5 multiple choice questions.

Probability of getting a correct answer is, $p = \frac{1}{3}$

$\therefore q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

Clearly, X has a binomial distribution with $n = 5$ and $p = \frac{1}{3}$

$$\therefore P(X = x) = {}^nC_x q^{n-x} p^x$$

$$= {}^5C_x \left(\frac{2}{3}\right)^{5-x} \cdot \left(\frac{1}{3}\right)^x$$

P (guessing more than 4 correct answers) $= P(X \geq 4)$

$$= P(X = 4) + P(X = 5)$$

$$= {}^5C_4 \left(\frac{2}{3}\right) \cdot \left(\frac{1}{3}\right)^4 + {}^5C_5 \left(\frac{1}{3}\right)^5$$

$$= 5 \cdot \frac{2}{3} \cdot \frac{1}{81} + 1 \cdot \frac{1}{243}$$

$$= \frac{10}{243} + \frac{1}{243}$$

$$= \frac{11}{243}$$

Question 10:

A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will in a prize (a) at least once (b) exactly once (c) at least twice?

Answer 10:

Let X represent the number of winning prizes in 50 lotteries. The trials are Bernoulli trials.

Clearly, X has a binomial distribution with $n = 50$ and $p = \frac{1}{100}$

$$\therefore q = 1 - p = 1 - \frac{1}{100} = \frac{99}{100}$$

$$\therefore \mathrm{P}(\mathrm{X} = x) = {}^{n}\mathrm{C}_{x} q^{n-x} p^{x} = {}^{50}\mathrm{C}_{x}\left(\frac{99}{100}\right)^{50-x} \cdot \left(\frac{1}{100}\right)^{x}$$

(a) P (winning at least once) = P (X ≥ 1)

$$= 1 - \mathrm{P}(\mathrm{X} < 1)$$
$$= 1 - \mathrm{P}(\mathrm{X} = 0)$$
$$= 1 - {}^{50}\mathrm{C}_{0}\left(\frac{99}{100}\right)^{50}$$
$$= 1 - 1 \cdot \left(\frac{99}{100}\right)^{50}$$
$$= 1 - \left(\frac{99}{100}\right)^{50}$$

(b) P (winning exactly once) = P(X = 1)

$$= {}^{50}\mathrm{C}_{1}\left(\frac{99}{100}\right)^{49} \cdot \left(\frac{1}{100}\right)^{1}$$
$$= 50\left(\frac{1}{100}\right)\left(\frac{99}{100}\right)^{49}$$
$$= \frac{1}{2}\left(\frac{99}{100}\right)^{49}$$

(c) P (at least twice) = P(X ≥ 2)

$$= 1 - \mathrm{P}(\mathrm{X} < 2)$$
$$= 1 - \mathrm{P}(\mathrm{X} \le 1)$$
$$= 1 - [\mathrm{P}(\mathrm{X} = 0) + \mathrm{P}(\mathrm{X} = 1)]$$
$$= [1 - \mathrm{P}(\mathrm{X} = 0)] - \mathrm{P}(\mathrm{X} = 1)$$
$$= 1 - \left(\frac{99}{100}\right)^{50} - \frac{1}{2} \cdot \left(\frac{99}{100}\right)^{49}$$
$$= 1 - \left(\frac{99}{100}\right)^{49} \cdot \left[\frac{99}{100} + \frac{1}{2}\right]$$
$$= 1 - \left(\frac{99}{100}\right)^{49} \cdot \left(\frac{149}{100}\right)$$
$$= 1 - \left(\frac{149}{100}\right)\left(\frac{99}{100}\right)^{49}$$

Question 11:

Find the probability of getting 5 exactly twice in 7 throws of a die.

Answer 11:

The repeated tossing of a die are Bernoulli trials. Let X represent the number of times of getting 5 in 7 throws of the die.

Probability of getting 5 in a single throw of the die, $p = \frac{1}{6}$

$\therefore q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

Clearly, X has the probability distribution with $n = 7$ and $p = \frac{1}{6}$

$\therefore \mathrm{P}(\mathrm{X} = x) = {}^{n}\mathrm{C}_{x} q^{n-x} p^{x} = {}^{7}\mathrm{C}_{x} \left(\frac{5}{6}\right)^{7-x} \cdot \left(\frac{1}{6}\right)^{x}$

P (getting 5 exactly twice) $= \mathrm{P}(\mathrm{X} = 2)$

$= {}^{7}\mathrm{C}_{2} \left(\frac{5}{6}\right)^{5} \cdot \left(\frac{1}{6}\right)^{2}$

$= 21 \cdot \left(\frac{5}{6}\right)^{5} \cdot \frac{1}{36}$

$= \left(\frac{7}{12}\right)\left(\frac{5}{6}\right)^{5}$

Question 12:

Find the probability of throwing at most 2 sixes in 6 throws of a single die.

Answer 12:

The repeated tossing of the die are Bernoulli trials. Let X represent the number of times of getting sixes in 6 throws of the die.

Probability of getting six in a single throw of die, $p = \frac{1}{6}$

$\therefore q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

Clearly, X has a binomial distribution with $n = 6$

$\therefore \mathrm{P}(\mathrm{X} = x) = {}^{n}\mathrm{C}_{x} q^{n-x} p^{x} = {}^{6}\mathrm{C}_{x} \left(\frac{5}{6}\right)^{6-x} \cdot \left(\frac{1}{6}\right)^{x}$

$$\therefore P(X=x)={}^{n}C_{x}q^{n-x}p^{x}={}^{6}C_{x}\left(\frac{5}{6}\right)^{6-x}\cdot\left(\frac{1}{6}\right)^{x}$$

P (at most 2 sixes) = P(X ≤ 2)

$$=P(X=0)+P(X=1)+P(X=2)$$

$$={}^{6}C_{0}\left(\frac{5}{6}\right)^{6}+{}^{6}C_{1}\cdot\left(\frac{5}{6}\right)^{5}\cdot\left(\frac{1}{6}\right)+{}^{6}C_{2}\left(\frac{5^{4}}{6}\right)\cdot\left(\frac{1}{6}\right)^{2}$$

$$=1\cdot\left(\frac{5}{6}\right)^{6}+6\cdot\frac{1}{6}\cdot\left(\frac{5}{6}\right)^{5}+15.\frac{1}{36}\cdot\left(\frac{5}{6}\right)^{4}$$

$$=\left(\frac{5}{6}\right)^{6}+\left(\frac{5}{6}\right)^{5}+\frac{5}{12}\cdot\left(\frac{5}{6}\right)^{4}$$

$$=\left(\frac{5}{6}\right)^{4}\left[\left(\frac{5}{6}\right)^{2}+\left(\frac{5}{6}\right)+\left(\frac{5}{12}\right)\right]$$

$$=\left(\frac{5}{6}\right)^{4}\cdot\left[\frac{25}{36}+\frac{5}{6}+\frac{5}{12}\right]$$

$$=\left(\frac{5}{6}\right)^{4}\cdot\left[\frac{25+30+15}{36}\right]$$

$$=\frac{70}{36}\cdot\left(\frac{5}{6}\right)^{4}$$

$$=\frac{35}{18}\cdot\left(\frac{5}{6}\right)^{4}$$

Question 13:

It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such articles, 9 are defective?

Answer 13:

The repeated selections of articles in a random sample space are Bernoulli trails. Let X denote the number of times of selecting defective articles in a random sample space of 12 articles.

Clearly, X has a binomial distribution with $n = 12$ and $p = 10\% = \frac{10}{100} = \frac{1}{10}$

$$\therefore q=1-p=1-\frac{1}{10}=\frac{9}{10}$$

$$\therefore P(X=x)={}^{n}C_{x}q^{n-x}p^{x}={}^{12}C_{x}\left(\frac{9}{10}\right)^{12-x}\cdot\left(\frac{1}{10}\right)^{x}$$

P (selecting 9 defective articles) $={}^{12}C_{9}\left(\frac{9}{10}\right)^{3}\left(\frac{1}{10}\right)^{9}$

$$=220\cdot\frac{9^{3}}{10^{3}}\cdot\frac{1}{10^{9}}$$

$$=\frac{22\times9^{3}}{10^{11}}$$

Question 14:

In a box containing 100 bulbs, 10 are defective. The probability that out of a sample of 5 bulbs, none is defective is

(A) 10^{-1}

(B) $\left(\frac{1}{2}\right)^5$

(C) $\left(\frac{9}{10}\right)^5$

(D) $\frac{9}{10}$

Answer 14:

The repeated selections of defective bulbs from a box are Bernoulli trials. Let X denote the number of defective bulbs out of a sample of 5 bulbs.

Probability of getting a defective bulb, $p = \frac{10}{100} = \frac{1}{10}$

$\therefore q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$

Clearly, X has a binomial distribution with $n = 5$ and $p = \frac{1}{10}$

$\therefore P(X = x) = {}^nC_x q^{n-x} p^x = {}^5C_x \left(\frac{9}{10}\right)^{5-x} \left(\frac{1}{10}\right)^x$

P (none of the bulbs is defective) $= P(X = 0)$

$= {}^5C_0 \cdot \left(\frac{9}{10}\right)^5$

$= 1 \cdot \left(\frac{9}{10}\right)^5$

$= \left(\frac{9}{10}\right)^5$

The correct answer is C.

Question 15:

The probability that a student is not a swimmer is $\frac{1}{5}$. Then the probability that out of five students, four are swimmers is

(A) ${}^5C_4\left(\frac{4}{5}\right)^4 \frac{1}{5}$ (B) $\left(\frac{4}{5}\right)^4 \frac{1}{5}$

(C) ${}^5C_1 \frac{1}{5}\left(\frac{4}{5}\right)^4$ (D) None of these

Answer 15:

The repeated selection of students who are swimmers are Bernoulli trials. Let X denote the number of students, out of 5 students, who are swimmers.

Probability of students who are not swimmers, $q=\frac{1}{5}$

$\therefore p=1-q=1-\frac{1}{5}=\frac{4}{5}$

Clearly, X has a binomial distribution with $n=5$ and $p=\frac{4}{5}$

$$\mathrm{P}(\mathrm{X}=x)={}^{n}\mathrm{C}_{x}q^{n-x}p^{x}={}^{5}\mathrm{C}_{x}\left(\frac{1}{5}\right)^{5-x}\cdot\left(\frac{4}{5}\right)^{x}$$

P (four students are swimmers) $=\mathrm{P}(\mathrm{X}=4)={}^{5}\mathrm{C}_{4}\left(\frac{1}{5}\right)\cdot\left(\frac{4}{5}\right)^{4}$

Therefore, the correct answer is A.