Exercise 7.10

Question 1:

$$\int_0^1 \frac{x}{x^2+1}dx$$

Answer 1:

$$\int_0^1 \frac{x}{x^2+1}dx$$

Let $x^2+1=t \Rightarrow 2x\,dx=dt$

When x = 0, t = 1 and when x = 1, t = 2

$$\therefore \int_0^1 \frac{x}{x^2+1}dx=\frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$=\frac{1}{2}\left[\log|t|\right]_1^2$$

$$=\frac{1}{2}\left[\log 2-\log 1\right]$$

$$=\frac{1}{2}\log 2$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\,d\phi$$

Answer 2:

Let $I=\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\,d\phi=\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\cos\phi\,d\phi$

Also, let $\sin\phi=t \Rightarrow \cos\phi\,d\phi=dt$

When $\phi = 0,\ t = 0$ and when $\phi = \frac{\pi}{2},\ t = 1$

$$\therefore I = \int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt$$
$$= \int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right)dt$$
$$= \int_0^1 \left[t^{\frac{1}{2}}+t^{\frac{9}{2}}-2t^{\frac{5}{2}}\right]dt$$
$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+\frac{t^{\frac{11}{2}}}{\frac{11}{2}}-\frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$
$$= \frac{2}{3}+\frac{2}{11}-\frac{4}{7}$$
$$= \frac{154+42-132}{231}$$
$$= \frac{64}{231}$$

Question 3:

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Answer 3:

Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Also, let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

When $x = 0,\ \theta = 0$ and when $x = 1,\ \theta = \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\sin 2\theta\right)\sec^2\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} 2\theta\cdot\sec^2\theta\, d\theta$$
$$= 2\int_0^{\frac{\pi}{4}} \theta\cdot\sec^2\theta\, d\theta$$

Taking θ as first function and $\sec^2\theta$ as second function and integrating by parts,

we obtain

$$I = 2\left[\theta \int \sec^2\theta\, d\theta - \int\left\{\left(\frac{d}{dx}\theta\right)\int \sec^2\theta\, d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta - \int \tan\theta\, d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$$

$$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$$

$$= \frac{\pi}{2} - \log 2$$

Question 4:

$\int_0^2 x\sqrt{x+2}$ $\left(\text{Put } x+2=t^2\right)$

Answer 4:

$\int_0^2 x\sqrt{x+2}\,dx$

Let x + 2 = t^2 ⇒ dx = 2tdt

When x = 0, $t=\sqrt{2}$ and when x = 2, t = 2

$$\therefore \int_0^2 x\sqrt{x+2}\,dx = \int_{\sqrt{2}}^{2}\left(t^2-2\right)\sqrt{t^2}\,2t\,dt$$
$$= 2\int_{\sqrt{2}}^{2}\left(t^2-2\right)t^2\,dt$$
$$= 2\int_{\sqrt{2}}^{2}\left(t^4-2t^2\right)dt$$
$$= 2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^{2}$$
$$= 2\left[\frac{32}{5}-\frac{16}{3}-\frac{4\sqrt{2}}{5}+\frac{4\sqrt{2}}{3}\right]$$
$$= 2\left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15}\right]$$
$$= 2\left[\frac{16+8\sqrt{2}}{15}\right]$$
$$= \frac{16\left(2+\sqrt{2}\right)}{15}$$
$$= \frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

Question 5:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$$

Answer 5:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$$

Let cos x = t ⇒ −sinx dx = dt

When x = 0, t = 1 and when $x=\frac{\pi}{2}, t=0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = -\int_1^0 \frac{dt}{1+t^2}$$
$$= -\left[\tan^{-1} t\right]_1^0$$
$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$
$$= -\left[-\frac{\pi}{4}\right]$$
$$= \frac{\pi}{4}$$

Question 6:

$$\int_0^2 \frac{dx}{x+4-x^2}$$

Answer 6:

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-\left(x^2-x-4\right)}$$
$$= \int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)}$$
$$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$$
$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

Let $x-\frac{1}{2}=t$ So dx = dt

When $x=0,\ t=-\frac{1}{2}$ and when $x=2,\ t=\frac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}=\int_{-\frac{1}{2}}^{\frac{3}{2}}\frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2-t^2}$$

$$=\left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)}\log\frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$=\frac{1}{\sqrt{17}}\left[\log\frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}}-\frac{\log\frac{\sqrt{17}}{2}-\frac{1}{2}}{\log\frac{\sqrt{17}}{2}+\frac{1}{2}}\right]$$

$$=\frac{1}{\sqrt{17}}\left[\log\frac{\sqrt{17}+3}{\sqrt{17}-3}-\log\frac{\sqrt{17}-1}{\sqrt{17}+1}\right]$$

$$=\frac{1}{\sqrt{17}}\log\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{\left(5+\sqrt{17}\right)\left(5+\sqrt{17}\right)}{25-17}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{25+17+10\sqrt{17}}{8}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{42+10\sqrt{17}}{8}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right)$$

Question 7:

$$\int_{-1}^{1}\frac{dx}{x^2+2x+5}$$

Answer 7:

$$\int_{-1}^{1}\frac{dx}{x^2+2x+5}=\int_{-1}^{1}\frac{dx}{\left(x^2+2x+1\right)+4}=\int_{-1}^{1}\frac{dx}{\left(x+1\right)^2+\left(2\right)^2}$$

Let x + 1 = t ⇒ dx = dt

When x = −1, t = 0 and when x = 1, t = 2

$$\therefore \int_{-1}^{1}\frac{dx}{\left(x+1\right)^2+\left(2\right)^2}=\int_{0}^{2}\frac{dt}{t^2+2^2}$$

$$=\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$$

$$=\frac{1}{2}\tan^{-1}1-\frac{1}{2}\tan^{-1}0$$

$$=\frac{1}{2}\left(\frac{\pi}{4}\right)=\frac{\pi}{8}$$

Question 8:

$$\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Answer 8:

$$\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Let 2x = t ⇒ 2dx = dt

When x = 1, t = 2 and when x = 2, t = 4

$$\therefore \int_1^2\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx=\frac{1}{2}\int_2^4\left(\frac{2}{t}-\frac{2}{t^2}\right)e^t dt$$

$$=\int_2^4\left(\frac{1}{t}-\frac{1}{t^2}\right)e^t dt$$

Let $\frac{1}{t}=f(t)$

Then, $f'(t)=-\frac{1}{t^2}$

$$\Rightarrow \int_2^4\left(\frac{1}{t}-\frac{1}{t^2}\right)e^t dt=\int_2^4 e^t\left[f(t)+f'(t)\right]dt$$

$$=\left[e^t f(t)\right]_2^4$$

$$=\left[e^t\cdot\frac{2}{t}\right]_2^4$$

$$=\left[\frac{e^t}{t}\right]_2^4$$

$$=\frac{e^4}{4}-\frac{e^2}{2}$$

$$=\frac{e^2\left(e^2-2\right)}{4}$$

Question 9:

The value of the integral $\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$ is

A. 6

B. 0

C. 3

D. 4

Answer 9:

$$\text{Let } I=\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$$

Also, let $x=\sin\theta \Rightarrow dx=\cos\theta\, d\theta$

When $x=\frac{1}{3},\ \theta=\sin^{-1}\left(\frac{1}{3}\right)$ and when $x=1,\ \theta=\frac{\pi}{2}$

$$\Rightarrow I=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta-\sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta\,d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(1-\sin^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta\,d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^4\theta}\cos\theta\,d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^2\theta\sin^2\theta}\cos\theta\,d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\cos\theta\right)^{\frac{5}{3}}}{\left(\sin\theta\right)^{\frac{5}{3}}}\operatorname{cosec}^2\theta\,d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\left(\cot\theta\right)^{\frac{5}{3}}\operatorname{cosec}^2\theta\,d\theta$$

Let $\cot\theta = t \Rightarrow -\operatorname{cosec}^2\theta\, d\theta = dt$

When $\theta=\sin^{-1}\left(\frac{1}{3}\right),\ t=2\sqrt{2}$ and when $\theta=\frac{\pi}{2},\ t=0$

$$\therefore I=-\int_{2\sqrt{2}}^{0}(t)^{\frac{5}{3}}dt$$

$$=-\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$=-\frac{3}{8}\left[(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$=-\frac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]$$

$$=\frac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$

$$=\frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$

$$=\frac{3}{8}[16]$$

$$=3\times 2$$

$$=6$$

Hence, the correct Answer is A.

Question 10:

If $f(x)=\int_0^x t\sin t\,dt$, then $f'(x)$ is

A. cos x + x sin x

B. x sin x

C. x cos x

D. sin x + x cos x

Answer 10:

$$f(x)=\int_0^x t\sin t\,dt$$

Integrating by parts, we obtain

$$f(x)=t\int_0^x \sin t\,dt-\int_0^x\left\{\left(\frac{d}{dt}t\right)\int \sin t\,dt\right\}dt$$
$$=\left[t(-\cos t)\right]_0^x-\int_0^x(-\cos t)\,dt$$
$$=\left[-t\cos t+\sin t\right]_0^x$$
$$=-x\cos x+\sin x$$

$$\Rightarrow f'(x)=-\left[\{x(-\sin x)\}+\cos x\right]+\cos x$$
$$=x\sin x-\cos x+\cos x$$
$$=x\sin x$$

Hence, the correct Answer is B.