Exercise 7.4

Integrate the functions in Exercises 1 to 23.

Question 1: $\frac{3x^2}{x^6+1}$

Answer 1:

Let $x^3 = t$

$\therefore 3x^2\,dx = dt$

$$\Rightarrow \int \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1}$$
$$= \tan^{1} t + C$$
$$= \tan^{-1}\left(x^3\right) + C$$

Question 2: $\frac{1}{\sqrt{1+4x^2}}$

Answer 2:

Let $2x = t$

$\therefore 2dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$
$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right] + C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$
$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right| + C$$

Question 3: $\dfrac{1}{\sqrt{(2-x)^2+1}}$

Answer 3:

Let 2 − x = t

⇒ −dx = dt

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$$

$$= -\log\left|t+\sqrt{t^2+1}\right| + C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= -\log\left|2-x+\sqrt{(2-x)^2+1}\right| + C$$

$$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right| + C$$

Question 4: $\dfrac{1}{\sqrt{9-25x^2}}$

Answer 4:

Let 5x = t

∴ 5dx = dt

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}\,dx = \frac{1}{5}\int \frac{1}{9-t^2}\,dt$$

$$= \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}}\,dt$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right) + C$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

Question 5: $\dfrac{3x}{1+2x^4}$

Answer 5:

Let $\sqrt{2}x^2 = t$

$\therefore\ 2\sqrt{2}x\ dx = dt$

$$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$
$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1} t\right] + C$$
$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

Question 6: $\dfrac{x^2}{1-x^6}$

Answer 6:

Let $x^3 = t$

$\therefore$ $3x^2$ dx = dt

$$\Rightarrow \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2}$$
$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$
$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

Question 7: $\dfrac{x-1}{\sqrt{x^2-1}}$

Answer 7:

$$\int \frac{x-1}{\sqrt{x^2-1}}\,dx = \int \frac{x}{\sqrt{x^2-1}}\,dx - \int \frac{1}{\sqrt{x^2-1}}\,dx \qquad \ldots(1)$$

For $\int \frac{x}{\sqrt{x^2-1}}dx$, let $x^2-1=t \Rightarrow 2x\,dx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\int t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$

$$= \sqrt{t}$$

$$= \sqrt{x^2-1}$$

From (1), we obtain

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \qquad \left[\int \frac{1}{\sqrt{x^2-a^2}}dt = \log\left|x+\sqrt{x^2-a^2}\right|\right]$$

$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

Question 8: $\dfrac{x^2}{\sqrt{x^6+a^6}}$

Answer 8:

Let $x^3 = t$

$\Rightarrow 3x^2\,dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$

$$= \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right| + C$$

$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C$$

Question 9: $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

Answer 9:

Let tan x = t

∴ $\sec^2 x$ dx = dt

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx = \int \frac{dt}{\sqrt{t^2+2^2}}$$
$$= \log\left|t+\sqrt{t^2+4}\right|+C$$
$$= \log\left|\tan x+\sqrt{\tan^2 x+4}\right|+C$$

Question 10: $\frac{1}{\sqrt{x^2+2x+2}}$

Answer 10:

$$\int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{(x+1)^2+(1)^2}}dx$$

Let $x+1=t$

$\therefore\ dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{t^2+1}}dt$$
$$= \log\left|t+\sqrt{t^2+1}\right|+C$$
$$= \log\left|(x+1)+\sqrt{(x+1)^2+1}\right|+C$$
$$= \log\left|(x+1)+\sqrt{x^2+2x+2}\right|+C$$

Question 11: $\dfrac{1}{\sqrt{9x^2+6x+5}}$

Answer 11:

$$\int\frac{1}{9x^2+6x+5}dx=\int\frac{1}{(3x+1)^2+(2)^2}dx$$

Let $(3x+1)=t$

$\therefore\ 3dx=dt$

$$\Rightarrow \int\frac{1}{(3x+1)^2+(2)^2}dx=\frac{1}{3}\int\frac{1}{t^2+2^2}dt$$

$$=\frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]+C$$

$$=\frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$$

Question 12: $\dfrac{1}{\sqrt{7-6x-x^2}}$

Answer 12:

$7-6x-x^2$ can be written as $7-\left(x^2+6x+9-9\right)$.

Therefore,

$7-\left(x^2+6x+9-9\right)$

$=16-\left(x^2+6x+9\right)$

$=16-(x+3)^2$

$=(4)^2-(x+3)^2$

$$\therefore\ \int\frac{1}{\sqrt{7-6x-x^2}}dx=\int\frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$$

Let $x+3=t$

$\Rightarrow\ dx=dt$

$$\Rightarrow \int\frac{1}{\sqrt{(4)^2-(x+3)^2}}dx=\int\frac{1}{\sqrt{(4)^2-(t)^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{4}\right)+C$$

$$=\sin^{-1}\left(\frac{x+3}{4}\right)+C$$

Question 13: $\dfrac{1}{\sqrt{(x-1)(x-2)}}$

Answer 13:

$(x-1)(x-2)$ can be written as x^2-3x+2.

Therefore,

$$x^2-3x+2$$

$$=x^2-3x+\frac{9}{4}-\frac{9}{4}+2$$

$$=\left(x-\frac{3}{2}\right)^2-\frac{1}{4}$$

$$=\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\therefore \int \frac{1}{\sqrt{(x-1)(x-2)}}dx=\int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

Let $x-\frac{3}{2}=t$

$\therefore\ dx=dt$

$$\Rightarrow \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx=\int \frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}\right|+C$$

$$=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$$

Question 14: $\frac{1}{\sqrt{8+3x-x^2}}$

Answer 14:

$8+3x-x^2$ can be written as $8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$.

Therefore,

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}}dx=\int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$$

Let $x-\frac{3}{2}=t$

$\therefore\ dx=dt$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx=\int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

Question 15: $\dfrac{1}{\sqrt{(x-a)(x-b)}}$

Answer 15:

$(x-a)(x-b)$ can be written as $x^2-(a+b)x+ab$.

Therefore,

$$x^2-(a+b)x+ab$$

$$=x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$$

$$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}$$

$$\Rightarrow\int\frac{1}{\sqrt{(x-a)(x-b)}}dx=\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx$$

Let $x-\left(\frac{a+b}{2}\right)=t$

$\therefore\ dx=dt$

$$\Rightarrow\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right|+C$$

$$=\log\left|\left\{x-\left(\frac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+C$$

Question 16: $\dfrac{4x+1}{\sqrt{2x^2+x-3}}$

Answer 16:

$$\text{Let } 4x+1 = A\frac{d}{dx}\left(2x^2+x-3\right)+B$$
$$\Rightarrow 4x+1 = A(4x+1)+B$$
$$\Rightarrow 4x+1 = 4Ax+A+B$$

Equating the coefficients of x and constant term on both sides, we obtain

4A = 4 ⇒ A = 1

A + B = 1 ⇒ B = 0

Let $2x^2 + x - 3 = t$

∴ (4x + 1) dx = dt

$$\Rightarrow \int\frac{4x+1}{\sqrt{2x^2+x-3}}dx = \int\frac{1}{\sqrt{t}}dt$$
$$= 2\sqrt{t}+C$$
$$= 2\sqrt{2x^2+x-3}+C$$

Question 17: $\dfrac{x+2}{\sqrt{x^2-1}}$

Answer 17:

$$\text{Let } x+2 = A\frac{d}{dx}\left(x^2-1\right)+B \qquad ...(1)$$
$$\Rightarrow x+2 = A(2x)+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$
$$B = 2$$

From (1), we obtain

$$(x+2)=\frac{1}{2}(2x)+2$$

$$\text{Then, } \int \frac{x+2}{\sqrt{x^2-1}}dx = \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx + \int \frac{2}{\sqrt{x^2-1}}dx \qquad ...(2)$$

$$\text{In } \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx, \text{ let } x^2-1=t \Rightarrow 2xdx=dt$$

$$\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\left[2\sqrt{t}\right]$$

$$= \sqrt{t}$$

$$= \sqrt{x^2-1}$$

$$\text{Then, } \int \frac{2}{\sqrt{x^2-1}}dx = 2\int \frac{1}{\sqrt{x^2-1}}dx = 2\log\left|x+\sqrt{x^2-1}\right|$$

From equation (2), we obtain

$$\int \frac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1}+2\log\left|x+\sqrt{x^2-1}\right|+C$$

Question 18: $\frac{5x-2}{1+2x+3x^2}$

Answer 18:

$$\text{Let } 5x-2 = A\frac{d}{dx}\left(1+2x+3x^2\right)+B$$

$$\Rightarrow 5x-2 = A(2+6x)+B$$

Equating the coefficient of x and constant term on both sides, we obtain

$5 = 6A \Rightarrow A = \frac{5}{6}$

$2A + B = -2 \Rightarrow B = -\frac{11}{3}$

$\therefore 5x - 2 = \frac{5}{6}(2+6x) + \left(-\frac{11}{3}\right)$

$$\Rightarrow \int \frac{5x-2}{1+2x+3x^2}dx = \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2}dx$$

$$= \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx - \frac{11}{3}\int \frac{1}{1+2x+3x^2}dx$$

Let $I_1 = \int \frac{2+6x}{1+2x+3x^2}dx$ and $I_2 = \int \frac{1}{1+2x+3x^2}dx$

$$\therefore \int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}I_1 - \frac{11}{3}I_2 \qquad ...(1)$$

$I_1 = \int \frac{2+6x}{1+2x+3x^2}dx$

Let $1 + 2x + 3x^2 = t$

$\Rightarrow (2+6x)dx = dt$

$\therefore I_1 = \int \frac{dt}{t}$

$I_1 = \log|t|$

$$I_1 = \log\left|1+2x+3x^2\right| \qquad ...(2)$$

$I_2 = \int \frac{1}{1+2x+3x^2}dx$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\frac{2}{3}x\right)$.

Therefore,

$$1+3\left(x^2+\frac{2}{3}x\right)$$

$$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$

$$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$$

$$=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2=\frac{1}{3}\int\frac{1}{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$

$$=\frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

$$=\frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad ...(3)$$

Substituting equations (2) and (3) in equation (1), we obtain

$$\int\frac{5x-2}{1+2x+3x^2}dx=\frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right]-\frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]+C$$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$

Question 19: $\dfrac{6x+7}{\sqrt{(x-5)(x-4)}}$

Answer 19:

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$$

$$\text{Let } 6x+7 = A\frac{d}{dx}\left(x^2-9x+20\right)+B$$

$$\Rightarrow 6x+7 = A(2x-9)+B$$

Equating the coefficients of x and constant term, we obtain

2A = 6 ⇒ A = 3

−9A + B = 7 ⇒ B = 34

∴ 6x + 7 = 3 (2x − 9) + 34

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}} = \int\frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

$$= 3\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx + 34\int\frac{1}{\sqrt{x^2-9x+20}}dx$$

$$\text{Let } I_1 = \int\frac{2x-9}{\sqrt{x^2-9x+20}}dx \text{ and } I_2 = \int\frac{1}{\sqrt{x^2-9x+20}}dx$$

$$\therefore \int\frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1+34I_2 \qquad ...(1)$$

Then,

$$I_1 = \int\frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

$$\text{Let } x^2-9x+20 = t$$

$$\Rightarrow (2x-9)dx = dt$$

$$\Rightarrow I_1 = \frac{dt}{\sqrt{t}}$$

$$I_1 = 2\sqrt{t}$$

$$I_1 = 2\sqrt{x^2-9x+20} \qquad ...(2)$$

$$\text{and } I_2 = \int\frac{1}{\sqrt{x^2-9x+20}}dx$$

$x^2-9x+20$ can be written as $x^2-9x+20+\frac{81}{4}-\frac{81}{4}$.

Therefore,

$$x^2-9x+20+\frac{81}{4}-\frac{81}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\frac{1}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2=\int\frac{1}{\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$$I_2=\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right| \qquad ...(3)$$

Substituting equations (2) and (3) in (1), we obtain

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}}dx=3\left[2\sqrt{x^2-9x+20}\right]+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

$$=6\sqrt{x^2-9x+20}+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

Question 20: $\frac{x+2}{\sqrt{4x-x^2}}$

Answer 20:

Let $x+2=A\frac{d}{dx}\left(4x-x^2\right)+B$

$$\Rightarrow x+2=A(4-2x)+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$-2A = 1 \Rightarrow A = -\frac{1}{2}$$
$$4A + B = 2 \Rightarrow B = 4$$
$$\Rightarrow (x+2) = -\frac{1}{2}(4-2x) + 4$$
$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$
$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{1}{\sqrt{4x-x^2}}dx$$

Let $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$ and $I_2 \int \frac{1}{\sqrt{4x-x^2}}dx$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}I_1 + 4I_2 \qquad ...(1)$$

Then, $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$

Let $4x - x^2 = t$

$$\Rightarrow (4-2x)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} \qquad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{4x-x^2}}dx$$
$$\Rightarrow 4x - x^2 = -(-4x + x^2)$$
$$= (-4x + x^2 + 4 - 4)$$
$$= 4 - (x-2)^2$$
$$= (2)^2 - (x-2)^2$$
$$\therefore I_2 = \int \frac{1}{\sqrt{(2)^2-(x-2)^2}}dx = \sin^{-1}\left(\frac{x-2}{2}\right) \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right)+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$
$$= -\sqrt{4x-x^2}+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

Question 21: $\frac{x+2}{\sqrt{x^2+2x+3}}$

Answer 21:

$$\int \frac{(x+2)}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \qquad ...(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x + 2)\, dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \qquad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$\Rightarrow x^2+2x+3 = x^2+2x+1+2 = (x+1)^2+\left(\sqrt{2}\right)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}dx = \log\left|(x+1)+\sqrt{x^2+2x+3}\right| \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\left[2\sqrt{x^2+2x+3}\right] + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

$$= \sqrt{x^2+2x+3} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

Question 22: $\frac{x+3}{x^2-2x-5}$

Answer 22:

$$\text{Let } (x+3) = A\frac{d}{dx}(x^2-2x-5)+B$$

$$(x+3) = A(2x-2)+B$$

Equating the coefficients of x and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$-2A+B = 3 \Rightarrow B = 4$$

$$\therefore (x+3) = \frac{1}{2}(2x-2)+4$$

$$\Rightarrow \int \frac{x+3}{x^2-2x-5}dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx + 4\int \frac{1}{x^2-2x-5}dx$$

$$\text{Let } I_1 = \int \frac{2x-2}{x^2-2x-5}dx \text{ and } I_2 = \int \frac{1}{x^2-2x-5}dx$$

$$\therefore \int \frac{x+3}{(x^2-2x-5)}dx = \frac{1}{2}I_1 + 4I_2 \qquad ...(1)$$

$$\text{Then, } I_1 = \int \frac{2x-2}{x^2-2x-5}dx$$

Let $x^2-2x-5=t$

$\Rightarrow (2x-2)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log\left|x^2-2x-5\right| \qquad ...(2)$$

$$I_2 = \int \frac{1}{x^2-2x-5}dx$$

$$= \int \frac{1}{(x^2-2x+1)-6}dx$$

$$= \int \frac{1}{(x-1)^2+\left(\sqrt{6}\right)^2}dx$$

$$= \frac{1}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) \qquad ...(3)$$

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

$$= \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

Question 23: $\dfrac{5x+3}{\sqrt{x^2+4x+10}}$

Answer 23:

Let $5x+3 = A\dfrac{d}{dx}\left(x^2+4x+10\right)+B$

$\Rightarrow 5x+3 = A(2x+4)+B$

Equating the coefficients of x and constant term, we obtain

$$2A = 5 \Rightarrow A = \frac{5}{2}$$

$$4A + B = 3 \Rightarrow B = -7$$

$$\therefore 5x + 3 = \frac{5}{2}(2x+4) - 7$$

$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

Let $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$

$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}I_1 - 7I_2 \qquad ...(1)$$

Then, $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$

Let $x^2 + 4x + 10 = t$

$$\therefore (2x+4)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} \qquad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= \int \frac{1}{\sqrt{(x^2+4x+4)+6}}dx$$

$$= \int \frac{1}{(x+2)^2+(\sqrt{6})^2}dx$$

$$= \log\left|(x+2)\sqrt{x^2+4x+10}\right| \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

Question 24: $\int \frac{dx}{x^2+2x+2}$ equals

(A) $x \tan^{-1}(x+1) + C$

(B) $\tan^{-1}(x+1) + C$

(C) $(x+1)\tan^{-1}x + C$

(D) $\tan^{-1}x + C$

Answer 24:

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$

$$= \int \frac{1}{(x+1)^2+(1)^2}\,dx$$

$$= \left[\tan^{-1}(x+1)\right] + C$$

Hence, the correct Answer is B.

Question 25: $\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

(B) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

(D) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Answer 25:

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$

$$= \int \frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$

$$= \int \frac{1}{-4\left(x^2-\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}\right)}dx$$

$$= \int \frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}dx$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right]+C \qquad \left(\int \frac{dy}{\sqrt{a^2-y^2}} = \sin^{-1}\frac{y}{a}+C\right)$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Hence, the correct Answer is B.