Exercise 7.8

Question 1:

$$\int_a^b x\, dx$$

Answer 1:

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\Big[f(a)+f(a+h)+\ldots+f\big(a+(n-1)h\big)\Big], \text{ where } h=\frac{b-a}{n}$$

Here, $a=a, b=b$, and $f(x)=x$

$$\therefore \int_a^b x\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\Big[a+(a+h)\ldots(a+2h)\ldots a+(n-1)h\Big]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\Big[\underbrace{(a+a+a+\ldots+a)}_{n \text{ times}}+\big(h+2h+3h+\ldots+(n-1)h\big)\Big]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\Big[na+h\big(1+2+3+\ldots+(n-1)\big)\Big]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+\frac{n(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{n}{n}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)(b-a)}{2n}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{\left(1-\frac{1}{n}\right)(b-a)}{2}\right]$$

$$=(b-a)\left[a+\frac{(b-a)}{2}\right]$$

$$=(b-a)\left[\frac{2a+b-a}{2}\right]$$

$$=\frac{(b-a)(b+a)}{2}$$

$$=\frac{1}{2}\big(b^2-a^2\big)$$

Question 2:

$$\int_0^5 (x+1)\,dx$$

Answer 2:

Let $I = \int_0^5 (x+1)\,dx$

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\Big[f(a)+f(a+h)\ldots f(a+(n-1)h)\Big], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 0$, $b = 5$, and $f(x) = (x+1)$

$$\Rightarrow h = \frac{5-0}{n} = \frac{5}{n}$$

$$\therefore \int_0^5 (x+1)\,dx = (5-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{5}{n}\right)+\ldots+f\left((n-1)\frac{5}{n}\right)\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{5}{n}+1\right)+\ldots\left\{1+\left(\frac{5(n-1)}{n}\right)\right\}\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[\underbrace{(1+1+1\ldots1)}_{n\text{ times}}+\left[\frac{5}{n}+2\cdot\frac{5}{n}+3\cdot\frac{5}{n}+\ldots(n-1)\frac{5}{n}\right]\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\{1+2+3\ldots(n-1)\}\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\cdot\frac{(n-1)n}{2}\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5(n-1)}{2}\right]$$

$$= 5\lim_{n\to\infty}\left[1+\frac{5}{2}\left(1-\frac{1}{n}\right)\right]$$

$$= 5\left[1+\frac{5}{2}\right]$$

$$= 5\left[\frac{7}{2}\right]$$

$$= \frac{35}{2}$$

Question 3:

$$\int_2^3 x^2 dx$$

Answer 3:

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f(a+2h)\ldots f\{a+(n-1)h\}\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=2$, $b=3$, and $f(x)=x^2$

$$\Rightarrow h=\frac{3-2}{n}=\frac{1}{n}$$

$$\therefore \int_2^3 x^2 dx = (3-2)\lim_{n\to\infty}\frac{1}{n}\left[f(2)+f\left(2+\frac{1}{n}\right)+f\left(2+\frac{2}{n}\right)\ldots f\left\{2+(n-1)\frac{1}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[(2)^2+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+\ldots\left(2+\frac{(n-1)}{n}\right)^2\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[2^2+\left\{2^2+\left(\frac{1}{n}\right)^2+2\cdot2\cdot\frac{1}{n}\right\}+\ldots+\left\{(2)^2+\frac{(n-1)^2}{n^2}+2\cdot2\cdot\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\left(\underset{n\ times}{2^2+\ldots+2^2}\right)+\left\{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+\left(\frac{n-1}{n}\right)^2\right\}+2\cdot2\cdot\left\{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\ldots+\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{1^2+2^2+3^2\ldots+(n-1)^2\right\}+\frac{4}{n}\left\{1+2+\ldots+(n-1)\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}+\frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{4n-4}{2}\right]$$

$$=\lim_{n\to\infty}\left[4+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+2-\frac{2}{n}\right]$$

$$=4+\frac{2}{6}+2$$

$$=\frac{19}{3}$$

Question 4:

$$\int_1^4 \left(x^2 - x\right) dx$$

Answer 4:

Let $I = \int_1^4 \left(x^2 - x\right) dx$

$$= \int_1^4 x^2 \, dx - \int_1^4 x \, dx$$

Let $I = I_1 - I_2$, where $I_1 = \int_1^4 x^2 dx$ and $I_2 = \int_1^4 x \, dx$ \quad ...(1)

It is known that,

$$\int_a^b f(x) dx = (b-a) \lim_{n \to \infty} \frac{1}{n} \left[f(a) + f(a+h) + f(a+(n-1)h) \right], \text{ where } h = \frac{b-a}{n}$$

For $I_1 = \int_1^4 x^2 dx$,

$a = 1, b = 4$, and $f(x) = x^2$

$$\therefore h = \frac{4-1}{n} = \frac{3}{n}$$

$$I_1 = \int_1^4 x^2 dx = (4-1) \lim_{n \to \infty} \frac{1}{n} \left[f(1) + f(1+h) + ... + f(1+(n-1)h) \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[1^2 + \left(1 + \frac{3}{n}\right)^2 + \left(1 + 2 \cdot \frac{3}{n}\right)^2 + ... \left(1 + \frac{(n-1)3}{n}\right)^2 \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[1^2 + \left\{ 1^2 + \left(\frac{3}{n}\right)^2 + 2 \cdot \frac{3}{n} \right\} + ... + \left\{ 1^2 + \left(\frac{(n-1)3}{n}\right)^2 + \frac{2 \cdot (n-1) \cdot 3}{n} \right\} \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[\left(\underset{n \text{ times}}{1^2 + ... + 1^2} \right) + \left(\frac{3}{n}\right)^2 \left\{ 1^2 + 2^2 + ... + (n-1)^2 \right\} + 2 \cdot \frac{3}{n} \left\{ 1 + 2 + ... + (n-1) \right\} \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[n + \frac{9}{n^2} \left\{ \frac{(n-1)(n)(2n-1)}{6} \right\} + \frac{6}{n} \left\{ \frac{(n-1)(n)}{2} \right\} \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[n + \frac{9n}{6} \left(1 - \frac{1}{n} \right) \left(2 - \frac{1}{n} \right) + \frac{6n-6}{2} \right]$$

$$= 3 \lim_{n \to \infty} \left[1 + \frac{9}{6} \left(1 - \frac{1}{n} \right) \left(2 - \frac{1}{n} \right) + 3 - \frac{3}{n} \right]$$

$$= 3 \left[1 + 3 + 3 \right]$$

$$= 3 \left[7 \right]$$

$$I_1 = 21 \qquad \qquad \qquad ...(2)$$

For $I_2 = \int_1^4 x\,dx,$

$a = 1, b = 4,$ and $f(x) = x$

$$\Rightarrow h = \frac{4-1}{n} = \frac{3}{n}$$

$$\therefore I_2 = (4-1) \lim_{n \to \infty} \frac{1}{n} \left[f(1) + f(1+h) + ...f(a+(n-1)h) \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[1 + (1+h) + ... + (1+(n-1)h) \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[1 + \left(1 + \frac{3}{n} \right) + ... + \left\{ 1 + (n-1)\frac{3}{n} \right\} \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[\underbrace{(1 + 1 + ... + 1)}_{n \text{ times}} + \frac{3}{n} \left(1 + 2 + ... + (n-1) \right) \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[n + \frac{3}{n} \left\{ \frac{(n-1)n}{2} \right\} \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[1 + \frac{3}{2} \left(1 - \frac{1}{n} \right) \right]$$

$$= 3 \left[1 + \frac{3}{2} \right]$$

$$= 3 \left[\frac{5}{2} \right]$$

$$I_2 = \frac{15}{2} \qquad \qquad \qquad ...(3)$$

From equations (2) and (3), we obtain

$$I = I_1 + I_2 = 21 - \frac{15}{2} = \frac{27}{2}$$

Question 5:

$$\int_{-1}^{1} e^x dx$$

Answer 5:

Let $I = \int_{-1}^{1} e^x dx$ ...(1)

It is known that,

$$\int_{a}^{b} f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)...f(a+(n-1)h)\right], \text{ where } h = \frac{b-a}{n}$$

Here, $a = -1$, $b = 1$, and $f(x) = e^x$

$$\therefore h = \frac{1+1}{n} = \frac{2}{n}$$

$$\therefore I = (1+1)\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2\cdot\frac{2}{n}\right)+...+f\left(-1+\frac{(n-1)2}{n}\right)\right]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2\frac{2}{n}\right)}+...e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1+e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+e^{(n-1)\frac{2}{n}}\right\}\right]$$

$$= 2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right]$$

$$= e^{-1}\times 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}}-1}\right]$$

$$= \frac{e^{-1}\times 2(e^2-1)}{\lim_{\frac{2}{n}\to 0}\left(\frac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right)\times 2}$$

$$= e^{-1}\left[\frac{2(e^2-1)}{2}\right] \qquad\qquad \left[\lim_{h\to 0}\left(\frac{e^h-1}{h}\right)=1\right]$$

$$= \frac{e^2-1}{e}$$

$$= \left(e-\frac{1}{e}\right)$$

Question 6:

$$\int_0^4 \left(x + e^{2x}\right) dx$$

Answer 6:

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\Big[f(a)+f(a+h)+\dots+f\big(a+(n-1)h\big)\Big], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 0$, $b = 4$, and $f(x) = x + e^{2x}$

$$\therefore h = \frac{4-0}{n} = \frac{4}{n}$$

$$\Rightarrow \int_0^4 \left(x + e^{2x}\right) dx = (4-0)\lim_{n\to\infty}\frac{1}{n}\Big[f(0)+f(h)+f(2h)+\dots+f\big((n-1)h\big)\Big]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\Big[\big(0+e^0\big)+\big(h+e^{2h}\big)+\big(2h+e^{2\cdot 2h}\big)+\dots+\big\{(n-1)h+e^{2(n-1)h}\big\}\Big]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\Big[1+\big(h+e^{2h}\big)+\big(2h+e^{4h}\big)+\dots+\big\{(n-1)h+e^{2(n-1)h}\big\}\Big]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\Big[\big\{h+2h+3h+\dots+(n-1)h\big\}+\big(1+e^{2h}+e^{4h}+\dots+e^{2(n-1)h}\big)\Big]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[h\big\{1+2+\dots(n-1)\big\}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{\big(h(n-1)n\big)}{2}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{4}{n}\cdot\frac{(n-1)n}{2}+\left(\frac{e^8-1}{e^{\frac{8}{n}}-1}\right)\right]$$

$$= 4(2)+4\lim_{n\to\infty}\frac{\big(e^8-1\big)}{\left(\frac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$$

$$= 8+\frac{4\cdot\big(e^8-1\big)}{8} \qquad\qquad \left(\lim_{x\to 0}\frac{e^x-1}{x}=1\right)$$

$$= 8+\frac{e^8-1}{2}$$

$$= \frac{15+e^8}{2}$$