Miscellaneous Solutions

Question 1:

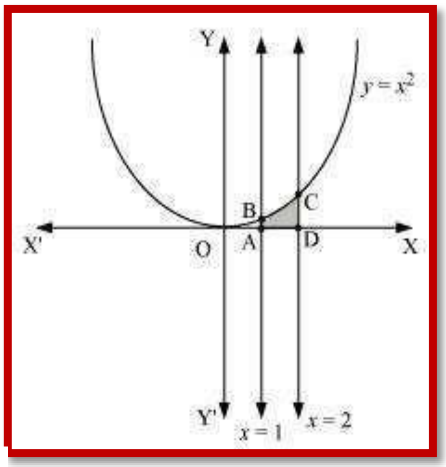
Find the area under the given curves and given lines:

(i) $y = x^2$, $x = 1$, $x = 2$ and x-axis

(ii) $y = x^4$, $x = 1$, $x = 5$ and x –axis

Answer 1:

i. The required area is represented by the shaded area ADCBA as



$$\text{Area ADCBA} = \int_1^2 y\,dx$$

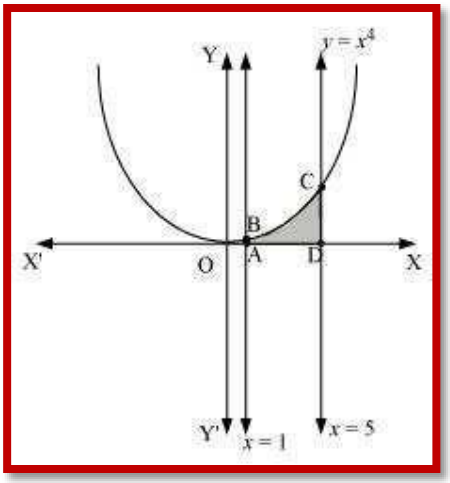
$$= \int_1^2 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{8}{3} - \frac{1}{3}$$

$$= \frac{7}{3} \text{ units}$$

ii. The required area is represented by the shaded area ADCBA as



$$\text{Area ADCBA} = \int_1^5 x^4 dx$$

$$= \left[\frac{x^5}{5}\right]_1^5$$

$$= \frac{(5)^5}{5} - \frac{1}{5}$$

$$= (5)^4 - \frac{1}{5}$$

$$= 625 - \frac{1}{5}$$

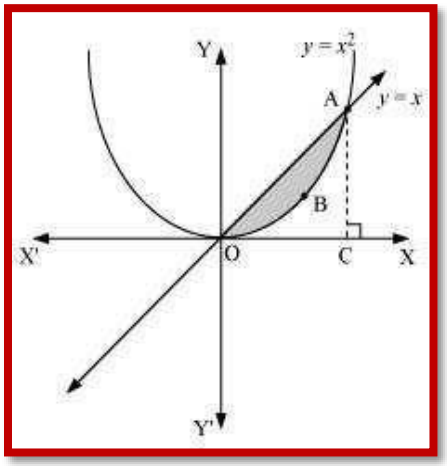
$$= 624.8 \text{ units}$$

Question 2:

Find the area between the curves y = x and y = x^2

Answer 2:

The required area is represented by the shaded area OBAO as



The points of intersection of the curves, y = x and y = x^2, is A (1, 1).

We draw AC perpendicular to x-axis.

∴ Area (OBAO) = Area (ΔOCA) – Area (OCABO) … (1)

$$= \int_0^1 x\,dx - \int_0^1 x^2\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

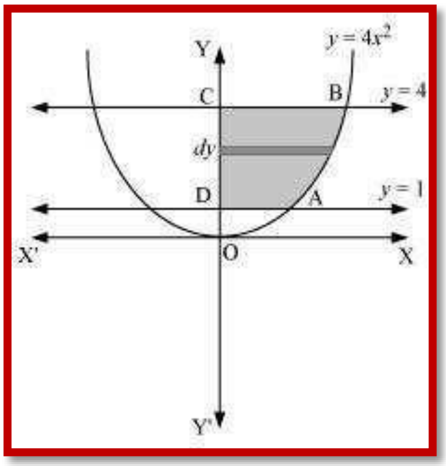
$$= \frac{1}{6} \text{ units}$$

Question 3:

Find the area of the region lying in the first quadrant and bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$

Answer 3:

The area in the first quadrant bounded by $y = 4x^2$, $x = 0$, $y = 1$, and $y = 4$ is represented by the shaded area ABCDA as



$$\therefore \text{Area ABCD} = \int_1^4 x\,dx$$

$$= \int_1^4 \frac{\sqrt{y}}{2} dx$$

$$= \frac{1}{2}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$

$$= \frac{1}{3}\left[(4)^{\frac{3}{2}} - 1\right]$$

$$= \frac{1}{3}[8-1]$$

$$= \frac{7}{3} \text{ units}$$

Question 4:

Sketch the graph of $y=|x+3|$ and evaluate $\int_{-6}^{0}|x+3|dx$

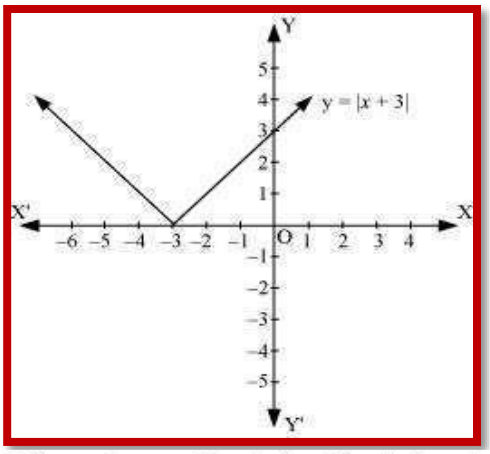
Answer 4:

The given equation is $y=|x+3|$

The corresponding values of x and y are given in the following table.

x	− 6	− 5	− 4	− 3	− 2	− 1	0
y	3	2	1	0	1	2	3

On plotting these points, we obtain the graph of $y=|x+3|$ as follows.



It is known that, $(x+3)\le 0$ for $-6\le x\le -3$ and $(x+3)\ge 0$ for $-3\le x\le 0$

$$\therefore \int_{-6}^{0}|(x+3)|dx = -\int_{-6}^{-3}(x+3)dx+\int_{-3}^{0}(x+3)dx$$

$$= -\left[\frac{x^2}{2}+3x\right]_{-6}^{-3}+\left[\frac{x^2}{2}+3x\right]_{-3}^{0}$$

$$= -\left[\left(\frac{(-3)^2}{2}+3(-3)\right)-\left(\frac{(-6)^2}{2}+3(-6)\right)\right]+\left[0-\left(\frac{(-3)^2}{2}+3(-3)\right)\right]$$

$$= -\left[-\frac{9}{2}\right]-\left[-\frac{9}{2}\right]$$

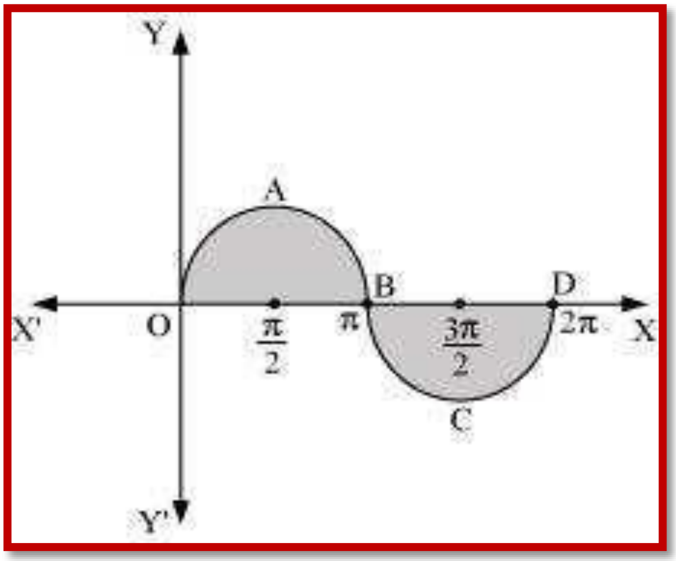
$$= 9$$

Question 5:

Find the area bounded by the curve y = sin x between x = 0 and x = 2π

Answer 5:

The graph of y = sin x can be drawn as



∴ Required area = Area OABO + Area BCDB

$$= \int_0^{\pi} \sin x\,dx + \left|\int_{\pi}^{2\pi} \sin x\,dx\right|$$

$$= [-\cos x]_0^{\pi} + \left|[-\cos x]_{\pi}^{2\pi}\right|$$

$$= [-\cos\pi + \cos 0] + |-\cos 2\pi + \cos\pi|$$

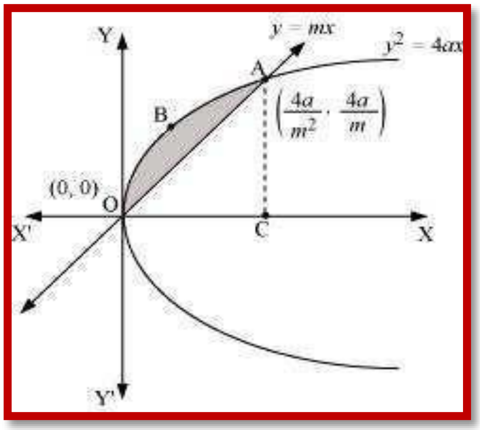
$$= 1 + 1 + |(-1-1)|$$

$$= 2 + |-2|$$

$$= 2 + 2 = 4 \text{ units}$$

Question 6:
Find the area enclosed between the parabola $y^2 = 4ax$ and the line $y = mx$

Answer 6:
The area enclosed between the parabola, $y^2 = 4ax$, and the line, $y = mx$, is represented by the shaded area OABO as



The points of intersection of both the curves are (0, 0) and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$.
We draw AC perpendicular to x-axis.

∴ Area OABO = Area OCABO – Area (ΔOCA)

$$= \int_0^{\frac{4a}{m^2}} 2\sqrt{ax}\,dx - \int_0^{\frac{4a}{m^2}} mx\,dx$$

$$= 2\sqrt{a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{4a}{m^2}} - m\left[\frac{x^2}{2}\right]_0^{\frac{4a}{m^2}}$$

$$= \frac{4}{3}\sqrt{a}\left(\frac{4a}{m^2}\right)^{\frac{3}{2}} - \frac{m}{2}\left[\left(\frac{4a}{m^2}\right)^2\right]$$

$$= \frac{32a^2}{3m^3} - \frac{m}{2}\left(\frac{16a^2}{m^4}\right)$$

$$= \frac{32a^2}{3m^3} - \frac{8a^2}{m^3}$$

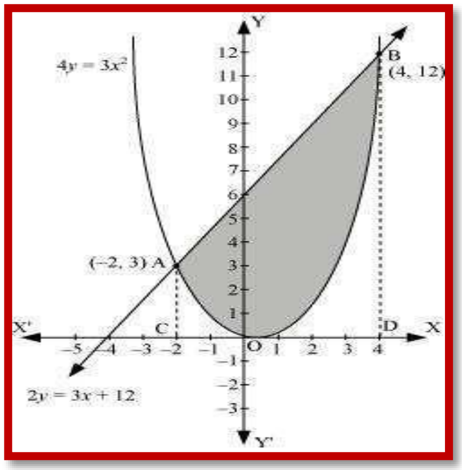
$$= \frac{8a^2}{3m^3} \text{ units}$$

Question 7:

Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$

Answer 7:

The area enclosed between the parabola, $4y = 3x^2$, and the line, $2y = 3x + 12$, is represented by the shaded area OBAO as



The points of intersection of the given curves are A (−2, 3) and (4, 12).

We draw AC and BD perpendicular to x-axis.

∴ Area OBAO = Area CDBA − (Area ODBO + Area OACO)

$$= \int_{-2}^{4} \frac{1}{2}(3x+12)\,dx - \int_{-2}^{4} \frac{3x^2}{4}\,dx$$

$$= \frac{1}{2}\left[\frac{3x^2}{2}+12x\right]_{-2}^{4} - \frac{3}{4}\left[\frac{x^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{2}[24+48-6+24] - \frac{1}{4}[64+8]$$

$$= \frac{1}{2}[90] - \frac{1}{4}[72]$$

$$= 45 - 18$$

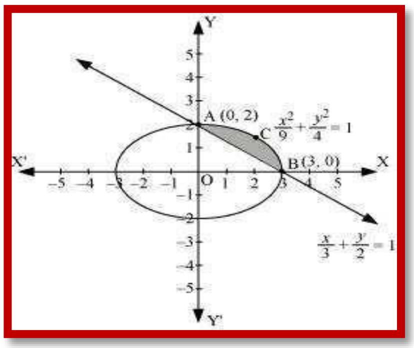
$$= 27 \text{ units}$$

Question 8:

Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$

Answer 8:

The area of the smaller region bounded by the ellipse, $\frac{x^2}{9}+\frac{y^2}{4}=1$, and the line, $\frac{x}{3}+\frac{y}{2}=1$, is represented by the shaded region BCAB



$\therefore$ Area BCAB = Area (OBCAO) – Area (OBAO)

$$=\int_0^3 2\sqrt{1-\frac{x^2}{9}}\,dx-\int_0^3 2\left(1-\frac{x}{3}\right)dx$$

$$=\frac{2}{3}\left[\int_0^3\sqrt{9-x^2}\,dx\right]-\frac{2}{3}\int_0^3(3-x)\,dx$$

$$=\frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3-\frac{2}{3}\left[3x-\frac{x^2}{2}\right]_0^3$$

$$=\frac{2}{3}\left[\frac{9}{2}\left(\frac{\pi}{2}\right)\right]-\frac{2}{3}\left[9-\frac{9}{2}\right]$$

$$=\frac{2}{3}\left[\frac{9\pi}{4}-\frac{9}{2}\right]$$

$$=\frac{2}{3}\times\frac{9}{4}(\pi-2)$$

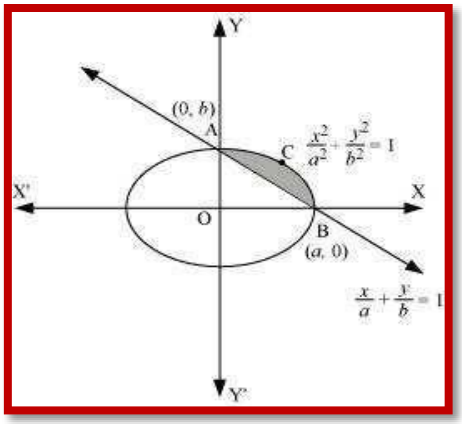
$$=\frac{3}{2}(\pi-2)\ units$$

Question 9:

Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$

Answer 9:

The area of the smaller region bounded by the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the line, $\frac{x}{a}+\frac{y}{b}=1$, is represented by the shaded region BCAB as



∴ Area BCAB = Area (OBCAO) – Area (OBAO)

$$= \int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx - \int_0^a b\left(1-\frac{x}{a}\right)dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx - \frac{b}{a}\int_0^a (a-x)dx$$

$$= \frac{b}{a}\left[\left\{\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\}_0^a - \left\{ax-\frac{x^2}{2}\right\}_0^a\right]$$

$$= \frac{b}{a}\left[\left\{\frac{a^2}{2}\left(\frac{\pi}{2}\right)\right\}-\left\{a^2-\frac{a^2}{2}\right\}\right]$$

$$= \frac{b}{a}\left[\frac{a^2\pi}{4}-\frac{a^2}{2}\right]$$

$$= \frac{ba^2}{2a}\left[\frac{\pi}{2}-1\right]$$

$$= \frac{ab}{2}\left[\frac{\pi}{2}-1\right]$$

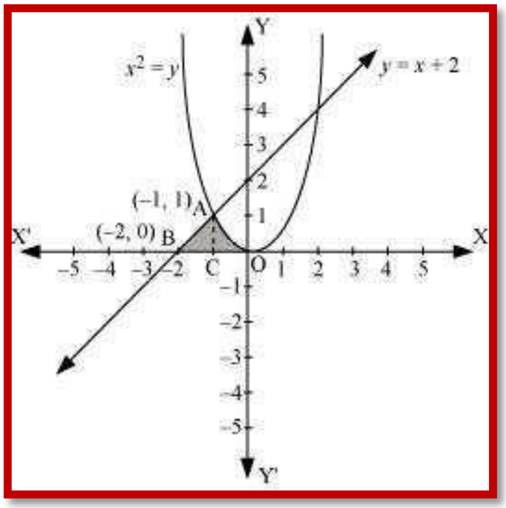
$$= \frac{ab}{4}(\pi-2)$$

Question 10:

Find the area of the region enclosed by the parabola $x^2 = y$, the line $y = x + 2$ and xaxis

Answer 10:

The area of the region enclosed by the parabola, $x^2 = y$, the line, $y = x + 2$, and x-axis is represented by the shaded region OABCO as



The point of intersection of the parabola, $x^2 = y$, and the line, $y = x + 2$, is A (−1, 1). ∴
Area OABCO = Area (BCA) + Area COAC

$$= \int_{-2}^{-1}(x+2)dx + \int_{-1}^{0} x^2 dx$$

$$= \left[\frac{x^2}{2} + 2x\right]_{-2}^{-1} + \left[\frac{x^3}{3}\right]_{-1}^{0}$$

$$= \left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-2)^2}{2} - 2(-2)\right] + \left[-\frac{(-1)^3}{3}\right]$$

$$= \left[\frac{1}{2} - 2 - 2 + 4 + \frac{1}{3}\right]$$

$$= \frac{5}{6} \text{ units}$$

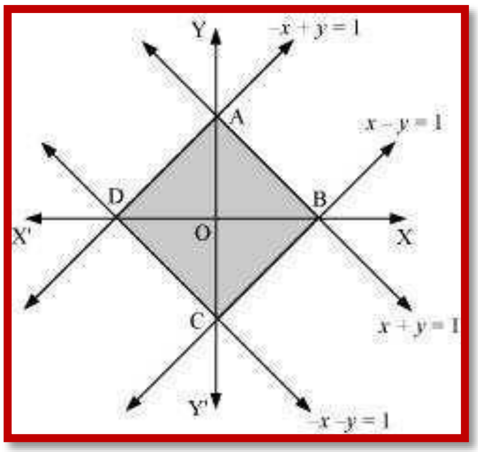
Question 11:

Using the method of integration find the area bounded by the curve $|x|+|y|=1$

[Hint: the required region is bounded by lines x + y = 1, x – y = 1, – x + y = 1 and – x – y = 11]

Answer 11:

The area bounded by the curve, $|x|+|y|=1$, is represented by the shaded region ADCB as



The curve intersects the axes at points A (0, 1), B (1, 0), C (0, −1), and D (−1, 0).

It can be observed that the given curve is symmetrical about x-axis and y-axis.

∴ Area ADCB = 4 × Area OBAO

$$= 4\int_0^1 (1-x)dx$$

$$= 4\left(x-\frac{x^2}{2}\right)_0^1$$

$$= 4\left[1-\frac{1}{2}\right]$$

$$= 4\left(\frac{1}{2}\right)$$

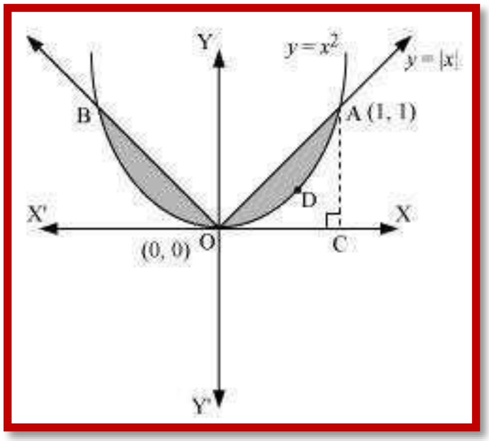
$$= 2 \text{ units}$$

Question 12:

Find the area bounded by curves $\{(x, y) : y \geq x^2 \text{ and } y = |x|\}$

Answer 12:

The area bounded by the curves, $\{(x, y) : y \geq x^2 \text{ and } y = |x|\}$, is represented by the shaded region as



It can be observed that the required area is symmetrical about y-axis.

$$\text{Required area} = 2\left[\text{Area}(\text{OCAO}) - \text{Area}(\text{OCADO})\right]$$

$$= 2\left[\int_0^1 x\,dx - \int_0^1 x^2\,dx\right]$$

$$= 2\left[\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right]$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

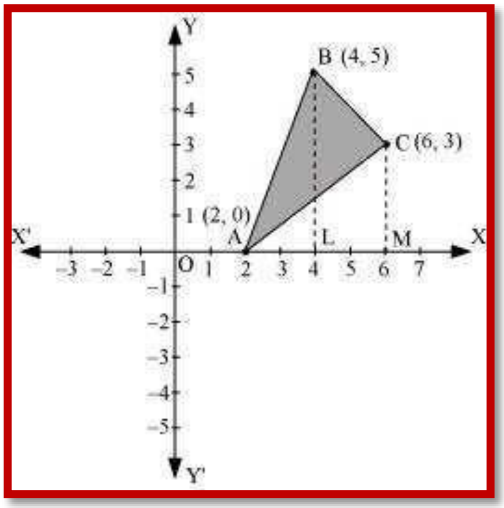
$$= 2\left[\frac{1}{6}\right] = \frac{1}{3} \text{ units}$$

Question 13:

Using the method of integration find the area of the triangle ABC, coordinates of whose vertices are A (2, 0), B (4, 5) and C (6, 3)

Answer 13:

The vertices of ΔABC are A (2, 0), B (4, 5), and C (6, 3).



Equation of line segment AB is

$$y-0=\frac{5-0}{4-2}(x-2)$$

$$2y=5x-10$$

$$y=\frac{5}{2}(x-2) \qquad \ldots(1)$$

Equation of line segment BC is

$$y-5=\frac{3-5}{6-4}(x-4)$$

$$2y-10=-2x+8$$

$$2y=-2x+18$$

$$y=-x+9 \qquad \ldots(2)$$

Equation of line segment CA is

$$y-3=\frac{0-3}{2-6}(x-6)$$
$$-4y+12=-3x+18$$
$$4y=3x-6$$
$$y=\frac{3}{4}(x-2) \qquad \ldots(3)$$

Area (ΔABC) = Area (ABLA) + Area (BLMCB) – Area (ACMA)

$$=\int_2^4 \frac{5}{2}(x-2)\,dx+\int_4^6(-x+9)\,dx-\int_2^6\frac{3}{4}(x-2)\,dx$$
$$=\frac{5}{2}\left[\frac{x^2}{2}-2x\right]_2^4+\left[\frac{-x^2}{2}+9x\right]_4^6-\frac{3}{4}\left[\frac{x^2}{2}-2x\right]_2^6$$
$$=\frac{5}{2}[8-8-2+4]+[-18+54+8-36]-\frac{3}{4}[18-12-2+4]$$
$$=5+8-\frac{3}{4}(8)$$
$$=13-6$$
$$=7 \text{ units}$$

Question 14:

Using the method of integration find the area of the region bounded by lines:

2x + y = 4, 3x – 2y = 6 and x – 3y + 5 = 0

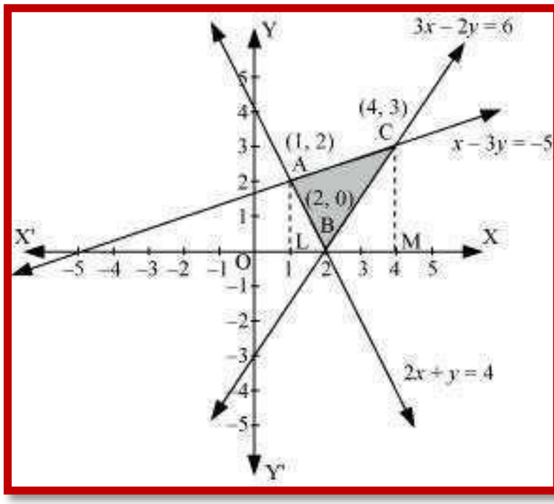
Answer 14:

The given equations of lines are

2x + y = 4 … (1)

3x – 2y = 6 … (2)

And, x – 3y + 5 = 0 … (3)



The area of the region bounded by the lines is the area of ΔABC. AL and CM are the perpendiculars on x-axis.

Area (ΔABC) = Area (ALMCA) − Area (ALB) − Area (CMB)

$$= \int_1^4 \left(\frac{x+5}{3}\right)dx - \int_1^2 (4-2x)dx - \int_2^4 \left(\frac{3x-6}{2}\right)dx$$

$$= \frac{1}{3}\left[\frac{x^2}{2}+5x\right]_1^4 - \left[4x-x^2\right]_1^2 - \frac{1}{2}\left[\frac{3x^2}{2}-6x\right]_2^4$$

$$= \frac{1}{3}\left[8+20-\frac{1}{2}-5\right] - [8-4-4+1] - \frac{1}{2}[24-24-6+12]$$

$$= \left(\frac{1}{3}\times\frac{45}{2}\right) - (1) - \frac{1}{2}(6)$$

$$= \frac{15}{2} - 1 - 3$$

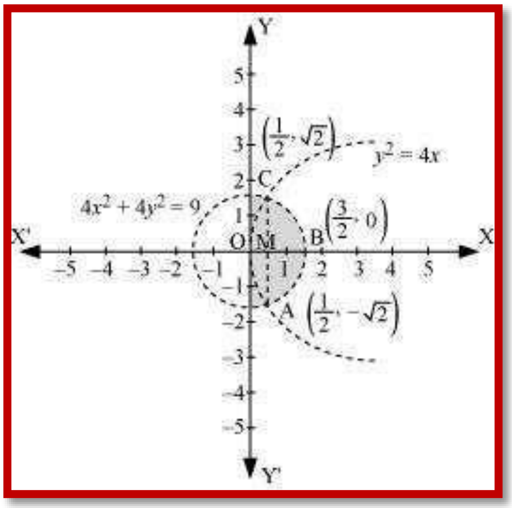
$$= \frac{15}{2} - 4 = \frac{15-8}{2} = \frac{7}{2} \text{ units}$$

Question 15:

Find the area of the region $\left\{(x, y): y^2 \le 4x, 4x^2 + 4y^2 \le 9\right\}$

Answer 15:

The area bounded by the curves, $\left\{(x, y): y^2 \le 4x, 4x^2 + 4y^2 \le 9\right\}$, is represented as



The points of intersection of both the curves are $\left(\frac{1}{2}, \sqrt{2}\right)$ and $\left(\frac{1}{2}, -\sqrt{2}\right)$.

The required area is given by OABCO.

It can be observed that area OABCO is symmetrical about x-axis.

∴ Area OABCO = 2 × Area OBC

Area OBCO = Area OMC + Area MBC

$$= \int_0^{\frac{1}{2}} 2\sqrt{x}\, dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}\sqrt{9 - 4x^2}\, dx$$

$$= \int_0^{\frac{1}{2}} 2\sqrt{x}\, dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}\sqrt{(3)^2 - (2x)^2}\, dx$$

Question 16:

Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is

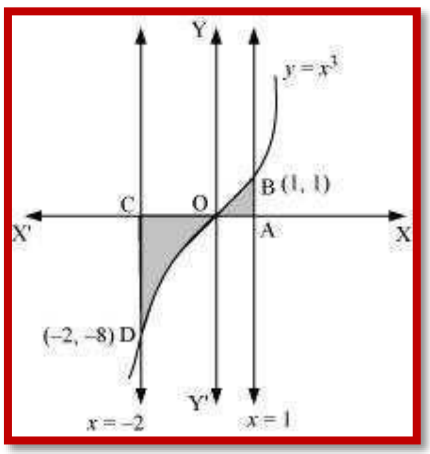
A. -9

B. $-\frac{15}{4}$

C. $\frac{15}{4}$

D. $\frac{17}{4}$

Answer 16:



Required area $= \int_{-2}^{1} y\,dx$

$= \int_{-2}^{1} x^3\,dx$

$= \left[\frac{x^4}{4}\right]_{-2}^{1}$

$= \left[\frac{1}{4} - \frac{(-2)^4}{4}\right]$

$= \left(\frac{1}{4} - 4\right) = -\frac{15}{4}$ units

Thus, the correct answer is B.

Question 17:

The area bounded by the curve $y = x|x|$, x-axis and the ordinates x = −1 and x = 1 is given by [Hint: $y = x^2$ if $x > 0$ and $y = -x^2$ if $x < 0$]

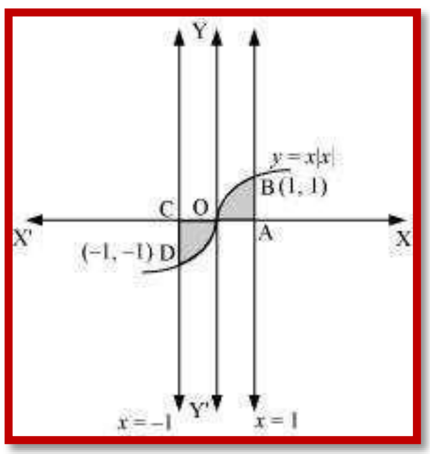
A. 0

B. $\frac{1}{3}$

C. $\frac{2}{3}$

D. $\frac{4}{3}$

Answer 17:



$$\text{Required area} = \int_{-1}^{1} y\,dx$$

$$= \int_{-1}^{1} x|x|\,dx$$

$$= \int_{-1}^{0} x^2\,dx + \int_{0}^{1} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{0}^{1}$$

$$= -\left(-\frac{1}{3}\right) + \frac{1}{3}$$

$$= \frac{2}{3} \text{ units}$$

Thus, the correct answer is C.

Question 18:

The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$ is

A. $\frac{4}{3}\left(4\pi - \sqrt{3}\right)$

B. $\frac{4}{3}\left(4\pi + \sqrt{3}\right)$

C. $\frac{4}{3}\left(8\pi - \sqrt{3}\right)$

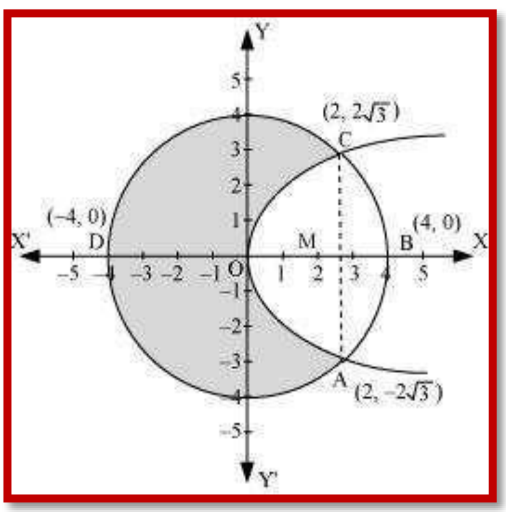
D. $\frac{4}{3}\left(4\pi + \sqrt{3}\right)$

Answer 18:

The given equations are

$x^2 + y^2 = 16$ … (1) $y^2 =$

$6x$ … (2)



Area bounded by the circle and parabola

$$= 2\left[\text{Area}(\text{OADO}) + \text{Area}(\text{ADBA})\right]$$

$$= 2\left[\int_0^2 \sqrt{16x}\,dx + \int_2^4 \sqrt{16 - x^2}\,dx\right]$$

$$= 2\left[\sqrt{6}\left\{\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right\}_0^2\right] + 2\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$= 2\sqrt{6} \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^2 + 2\left[8 \cdot \frac{\pi}{2} - \sqrt{16 - 4} - 8\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \frac{4\sqrt{6}}{3}\left(2\sqrt{2}\right) + 2\left[4\pi - \sqrt{12} - 8\frac{\pi}{6}\right]$$

$$= \frac{16\sqrt{3}}{3} + 8\pi - 4\sqrt{3} - \frac{8}{3}\pi$$

$$= \frac{4}{3}\left[4\sqrt{3} + 6\pi - 3\sqrt{3} - 2\pi\right]$$

$$= \frac{4}{3}\left[\sqrt{3} + 4\pi\right]$$

$$= \frac{4}{3}\left[4\pi + \sqrt{3}\right] \text{ units}$$

Area of circle = $\pi (r)^2$

= $\pi (4)^2$

= 16π units

$$\therefore \text{Required area} = 16\pi - \frac{4}{3}\left[4\pi + \sqrt{3}\right]$$

$$= \frac{4}{3}\left[4 \times 3\pi - 4\pi - \sqrt{3}\right]$$

$$= \frac{4}{3}\left(8\pi - \sqrt{3}\right) \text{ units}$$

Thus, the correct answer is C.

Question 19:

The area bounded by the y-axis, y = cos x and y = sin x when $0 \le x \le \frac{\pi}{2}$

A. $2\left(\sqrt{2}-1\right)$

B. $\sqrt{2}-1$

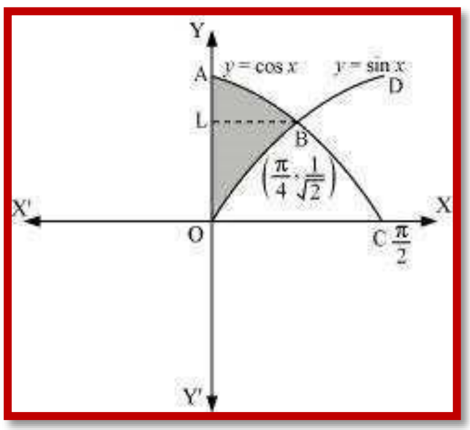
C. $\sqrt{2}+1$

D. $\sqrt{2}$

Answer 19:

The given equations are y = cos x … (1)

And, y = sin x … (2)



Required area = Area (ABLA) + area (OBLO)

$$= \int_{\frac{1}{\sqrt{2}}}^{1} x dy + \int_{0}^{\frac{1}{\sqrt{2}}} x dy$$

$$= \int_{\frac{1}{\sqrt{2}}}^{1} \cos^{-1} y dy + \int_{0}^{\frac{1}{\sqrt{2}}} \sin^{-1} x dy$$

Integrating by parts, we obtain

$$= \left[y\cos^{-1} y - \sqrt{1-y^2} \right]_{\frac{1}{\sqrt{2}}}^{1} + \left[x\sin^{-1} x + \sqrt{1-x^2} \right]_{0}^{\frac{1}{\sqrt{2}}}$$

$$= \left[\cos^{-1}(1) - \frac{1}{\sqrt{2}}\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) + \sqrt{1-\frac{1}{2}} \right] + \left[\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \sqrt{1-\frac{1}{2}} - 1 \right]$$

$$= \frac{-\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1$$

$$= \frac{2}{\sqrt{2}} - 1$$

$$= \sqrt{2} - 1 \text{ units}$$

Thus, the correct answer is B.