Exercise 9.4

Question 1:

$$\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$$

Answer

The given differential equation is:

$$\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}=\tan^2\frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx}=\left(\sec^2\frac{x}{2}-1\right)$$

Separating the variables,we get:

$$dy=\left(\sec^2\frac{x}{2}-1\right)dx$$

Now, integrating both sides of this equation, we get:

$$\int dy=\int\left(\sec^2\frac{x}{2}-1\right)dx=\int\sec^2\frac{x}{2}dx-\int dx$$

$$\Rightarrow y=2\tan\frac{x}{2}-x+C$$

This is the required general solution of the given differential equation.

Question 2:

$$\frac{dy}{dx}=\sqrt{4-y^2}\ \left(-2<y<2\right)$$

Answer

The given differential equation is:

$$\frac{dy}{dx}=\sqrt{4-y^2}$$

Separating the variables, we get:

$$\Rightarrow \frac{dy}{\sqrt{4-y^2}}=dx$$

Now, integrating both sides of this equation, we get:

$$\int \frac{dy}{\sqrt{4-y^2}}=\int dx$$

$$\Rightarrow \sin^{-1}\frac{y}{2}=x+C$$

$$\Rightarrow \frac{y}{2}=\sin(x+C)$$

$$\Rightarrow y=2\sin(x+C)$$

This is the required general solution of the given differential equation.

Question 3:

$$\frac{dy}{dx}+y=1\ (y\neq 1)$$

Answer

The given differential equation is:

$$\frac{dy}{dx}+y=1$$

$$\Rightarrow dy+y\ dx=dx$$

$$\Rightarrow dy=(1-y)dx$$

Separating the variables, we get:

$$\Rightarrow \frac{dy}{1-y}=dx$$

Now, integrating both sides, we get:

$$\int \frac{dy}{1-y} = \int dx$$

$$\Rightarrow \log(1-y) = x + \log C$$

$$\Rightarrow -\log C - \log(1-y) = x$$

$$\Rightarrow \log C(1-y) = -x$$

$$\Rightarrow C(1-y) = e^{-x}$$

$$\Rightarrow 1-y = \frac{1}{C}e^{-x}$$

$$\Rightarrow y = 1-\frac{1}{C}e^{-x}$$

$$\Rightarrow y = 1 + Ae^{-x} \text{ (where } A = -\frac{1}{C}\text{)}$$

This is the required general solution of the given differential equation.

Question 4:

$$\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$$

Answer

The given differential equation is:

$$\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$$

$$\Rightarrow \frac{\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy}{\tan x \tan y} = 0$$

$$\Rightarrow \frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

$$\Rightarrow \frac{\sec^2 x}{\tan x}dx = -\frac{\sec^2 y}{\tan y}dy$$

Integrating both sides of this equation, we get:

$$\int \frac{\sec^2 x}{\tan x}dx = -\int \frac{\sec^2 y}{\tan y}dy \qquad \ldots(1)$$

Let $\tan x = t$.

$$\therefore \frac{d}{dx}(\tan x) = \frac{dt}{dx}$$

$$\Rightarrow \sec^2 x = \frac{dt}{dx}$$

$$\Rightarrow \sec^2 x \, dx = dt$$

$$\text{Now, } \int \frac{\sec^2 x}{\tan x} dx = \int \frac{1}{t} dt.$$

$$= \log t$$

$$= \log(\tan x)$$

$$\text{Similarly, } \int \frac{\sec^2 x}{\tan x} dy = \log(\tan y).$$

Substituting these values in equation (1), we get:

$$\log(\tan x) = -\log(\tan y) + \log C$$

$$\Rightarrow \log(\tan x) = \log\left(\frac{C}{\tan y}\right)$$

$$\Rightarrow \tan x = \frac{C}{\tan y}$$

$$\Rightarrow \tan x \tan y = C$$

This is the required general solution of the given differential equation.

Question 5:

$$\left(e^x + e^{-x}\right)dy - \left(e^x - e^{-x}\right)dx = 0$$

Answer

The given differential equation is:

$$\left(e^x + e^{-x}\right)dy - \left(e^x - e^{-x}\right)dx = 0$$

$$\Rightarrow \left(e^x + e^{-x}\right)dy = \left(e^x - e^{-x}\right)dx$$

$$\Rightarrow dy = \left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]dx$$

Integrating both sides of this equation, we get:

$$\int dy = \int\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]dx + \mathrm{C}$$

$$\Rightarrow y = \int\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]dx + \mathrm{C} \qquad ...(1)$$

Let $(e^x + e^{-x}) = t$.

Differentiating both sides with respect to x, we get:

$$\frac{d}{dx}\left(e^x + e^{-x}\right) = \frac{dt}{dx}$$

$$\Rightarrow e^x - e^{-x} = \frac{dt}{dt}$$

$$\Rightarrow \left(e^x - e^{-x}\right)dx = dt$$

Substituting this value in equation (1), we get:

$$y = \int\frac{1}{t}dt + \mathrm{C}$$

$$\Rightarrow y = \log(t) + \mathrm{C}$$

$$\Rightarrow y = \log\left(e^x + e^{-x}\right) + \mathrm{C}$$

This is the required general solution of the given differential equation.

Question 6:

$$\frac{dy}{dx} = \left(1 + x^2\right)\left(1 + y^2\right)$$

Answer

The given differential equation is:

$$\frac{dy}{dx} = \left(1 + x^2\right)\left(1 + y^2\right)$$

$$\Rightarrow \frac{dy}{1 + y^2} = \left(1 + x^2\right)dx$$

Integrating both sides of this equation, we get:

$$\int \frac{dy}{1+y^2} = \int \left(1+x^2\right)dx$$

$$\Rightarrow \tan^{-1} y = \int dx + \int x^2 dx$$

$$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + C$$

This is the required general solution of the given differential equation.

Question 7:

$$y \log y \, dx - x \, dy = 0$$

Answer

The given differential equation is:

$$y \log y \, dx - x \, dy = 0$$

$$\Rightarrow y \log y \, dx = x \, dy$$

$$\Rightarrow \frac{dy}{y \log y} = \frac{dx}{x}$$

Integrating both sides, we get:

$$\int \frac{dy}{y \log y} = \int \frac{dx}{x} \qquad \ldots(1)$$

Let $\log y = t$.

$$\therefore \frac{d}{dy}(\log y) = \frac{dt}{dy}$$

$$\Rightarrow \frac{1}{y} = \frac{dt}{dy}$$

$$\Rightarrow \frac{1}{y} dy = dt$$

Substituting this value in equation (1), we get:

$$\int\frac{dt}{t}=\int\frac{dx}{x}$$
$$\Rightarrow \log t=\log x+\log C$$
$$\Rightarrow \log(\log y)=\log Cx$$
$$\Rightarrow \log y=Cx$$
$$\Rightarrow y=e^{Cx}$$

This is the required general solution of the given differential equation.

Question 8:

$$x^5\frac{dy}{dx}=-y^5$$

Answer

The given differential equation is:

$$x^5\frac{dy}{dx}=-y^5$$
$$\Rightarrow \frac{dy}{y^5}=-\frac{dx}{x^5}$$
$$\Rightarrow \frac{dx}{x^5}+\frac{dy}{y^5}=0$$

Integrating both sides, we get:

$$\int\frac{dx}{x^5}+\int\frac{dy}{y^5}=k \quad \text{(where } k \text{ is any constant)}$$
$$\Rightarrow \int x^{-5}dx+\int y^{-5}dy=k$$
$$\Rightarrow \frac{x^{-4}}{-4}+\frac{y^{-4}}{-4}=k$$
$$\Rightarrow x^{-4}+y^{-4}=-4k$$
$$\Rightarrow x^{-4}+y^{-4}=C \qquad (C=-4k)$$

This is the required general solution of the given differential equation.

Question 9:

$$\frac{dy}{dx}=\sin^{-1}x$$

Answer

The given differential equation is:

$$\frac{dy}{dx}=\sin^{-1}x$$

$$\Rightarrow dy=\sin^{-1}x\,dx$$

Integrating both sides, we get:

$$\int dy=\int\sin^{-1}x\,dx$$

$$\Rightarrow y=\int\left(\sin^{-1}x\cdot 1\right)dx$$

$$\Rightarrow y=\sin^{-1}x\cdot\int(1)\,dx-\int\left[\left(\frac{d}{dx}\left(\sin^{-1}x\right)\cdot\int(1)\,dx\right)\right]dx$$

$$\Rightarrow y=\sin^{-1}x\cdot x-\int\left(\frac{1}{\sqrt{1-x^2}}\cdot x\right)dx$$

$$\Rightarrow y=x\sin^{-1}x+\int\frac{-x}{\sqrt{1-x^2}}dx \qquad ...(1)$$

Let $1-x^2=t$.

$$\Rightarrow\frac{d}{dx}\left(1-x^2\right)=\frac{dt}{dx}$$

$$\Rightarrow -2x=\frac{dt}{dx}$$

$$\Rightarrow x\,dx=-\frac{1}{2}dt$$

Substituting this value in equation (1), we get:

$y = x\sin^{-1}x + \int \frac{1}{2\sqrt{t}}dt$

$\Rightarrow y = x\sin^{-1}x + \frac{1}{2}\cdot\int (t)^{-\frac{1}{2}}dt$

$\Rightarrow y = x\sin^{-1}x + \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$

$\Rightarrow y = x\sin^{-1}x + \sqrt{t} + C$

$\Rightarrow y = x\sin^{-1}x + \sqrt{1-x^2} + C$

This is the required general solution of the given differential equation.

Question 10:

$e^x \tan y\, dx + (1-e^x)\sec^2 y\, dy = 0$

Answer

The given differential equation is:

$e^x \tan y\, dx + (1-e^x)\sec^2 y\, dy = 0$

$(1-e^x)\sec^2 y\, dy = -e^x \tan y\, dx$

Separating the variables, we get:

$\frac{\sec^2 y}{\tan y}dy = \frac{-e^x}{1-e^x}dx$

Integrating both sides, we get:

$$\int \frac{\sec^2 y}{\tan y}dy = \int \frac{-e^x}{1-e^x}dx \quad ...(1)$$

Let $\tan y = u$.

$\Rightarrow \frac{d}{dy}(\tan y) = \frac{du}{dy}$

$\Rightarrow \sec^2 y = \frac{du}{dy}$

$\Rightarrow \sec^2 y dy = du$

$\therefore \int \frac{\sec^2 y}{\tan y}dy = \int \frac{du}{u} = \log u = \log(\tan y)$

Now, let $1-e^x=t$.

$$\therefore \frac{d}{dx}\left(1-e^x\right)=\frac{dt}{dx}$$

$$\Rightarrow -e^x=\frac{dt}{dx}$$

$$\Rightarrow -e^x dx=dt$$

$$\Rightarrow \int\frac{-e^x}{1-e^x}dx=\int\frac{dt}{t}=\log t=\log\left(1-e^x\right)$$

Substituting the values of $\int\frac{\sec^2 y}{\tan y}dy$ and $\int\frac{-e^x}{1-e^x}dx$

$$\Rightarrow \log\left(\tan y\right)=\log\left(1-e^x\right)+\log C$$

$$\Rightarrow \log\left(\tan y\right)=\log\left[C\left(1-e^x\right)\right]$$

$$\Rightarrow \tan y=C\left(1-e^x\right)$$

This is the required general solution of the given differential equation.

Question 11:

$$\left(x^3+x^2+x+1\right)\frac{dy}{dx}=2x^2+x;\ y=1 \text{ when } x=0$$

Answer

The given differential equation is:

$$\left(x^3+x^2+x+1\right)\frac{dy}{dx}=2x^2+x$$

$$\Rightarrow \frac{dy}{dx}=\frac{2x^2+x}{\left(x^3+x^2+x+1\right)}$$

$$\Rightarrow dy=\frac{2x^2+x}{\left(x+1\right)\left(x^2+1\right)}dx$$

Integrating both sides, we get:

$$\int dy = \int \frac{2x^2+x}{(x+1)(x^2+1)}dx \qquad ...(1)$$

$$\text{Let } \frac{2x^2+x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}. \qquad ...(2)$$

$$\Rightarrow \frac{2x^2+x}{(x+1)(x^2+1)} = \frac{Ax^2+A+(Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$$\Rightarrow 2x^2+x = Ax^2+A+Bx^2+Bx+Cx+C$$

$$\Rightarrow 2x^2+x = (A+B)x^2+(B+C)x+(A+C)$$

Comparing the coefficients of x^2 and x, we get:

A + B = 2

B + C = 1

A + C = 0

Solving these equations, we get:

$$A = \frac{1}{2},\ B = \frac{3}{2} \text{ and } C = \frac{-1}{2}$$

Substituting the values of A, B, and C in equation (2), we get:

$$\frac{2x^2+x}{(x+1)(x^2+1)} = \frac{1}{2}\cdot\frac{1}{(x+1)} + \frac{1}{2}\frac{(3x-1)}{(x^2+1)}$$

Therefore, equation (1) becomes:

$$\int dy = \frac{1}{2}\int\frac{1}{x+1}dx + \frac{1}{2}\int\frac{3x-1}{x^2+1}dx$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{2}\int\frac{x}{x^2+1}dx - \frac{1}{2}\int\frac{1}{x^2+1}dx$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\cdot\int\frac{2x}{x^2+1}dx - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{4}\left[2\log(x+1) + 3\log(x^2+1)\right] - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{4}\left[(x+1)^2(x^2+1)^3\right] - \frac{1}{2}\tan^{-1}x + C \qquad \ldots(3)$$

Now, $y = 1$ when $x = 0$.

$$\Rightarrow 1 = \frac{1}{4}\log(1) - \frac{1}{2}\tan^{-1}0 + C$$

$$\Rightarrow 1 = \frac{1}{4}\times 0 - \frac{1}{2}\times 0 + C$$

$$\Rightarrow C = 1$$

Substituting C = 1 in equation (3), we get:

$$y = \frac{1}{4}\left[\log(x+1)^2(x^2+1)^3\right] - \frac{1}{2}\tan^{-1}x + 1$$

Question 12:

$$x(x^2-1)\frac{dy}{dx} = 1;\ y = 0 \text{ when } x = 2$$

Answer

$$x(x^2-1)\frac{dy}{dx} = 1$$

$$\Rightarrow dy = \frac{dx}{x(x^2-1)}$$

$$\Rightarrow dy = \frac{1}{x(x-1)(x+1)}dx$$

Integrating both sides, we get:

$$\int dy = \int \frac{1}{x(x-1)(x+1)} dx \qquad ...(1)$$

$$\text{Let } \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}. \qquad ...(2)$$

$$\Rightarrow \frac{1}{x(x-1)(x+1)} = \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$$= \frac{(A+B+C)x^2 + (B-C)x - A}{x(x-1)(x+1)}$$

Comparing the coefficients of x^2, x, and constant, we get:

$$A = -1$$
$$B - C = 0$$
$$A + B + C = 0$$

Solving these equations, we get

$$B = \frac{1}{2} \text{ and } C = \frac{1}{2}.$$

Substituting the values of A, B, and C in equation (2), we get:

$$\frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}$$

Therefore, equation (1) becomes:

$$\int dy = -\int \frac{1}{x}dx + \frac{1}{2}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{x+1}dx$$

$$\Rightarrow y = -\log x + \frac{1}{2}\log(x-1) + \frac{1}{2}\log(x+1) + \log k$$

$$\Rightarrow y = \frac{1}{2}\log\left[\frac{k^2(x-1)(x+1)}{x^2}\right] \qquad ...(3)$$

Now, $y = 0$ when $x = 2$.

$$\Rightarrow 0 = \frac{1}{2}\log\left[\frac{k^2(2-1)(2+1)}{4}\right]$$

$$\Rightarrow \log\left(\frac{3k^2}{4}\right) = 0$$

$$\Rightarrow \frac{3k^2}{4} = 1$$

$$\Rightarrow 3k^2 = 4$$

$$\Rightarrow k^2 = \frac{4}{3}$$

Substituting the value of k^2 in equation (3), we get:

$$y = \frac{1}{2}\log\left[\frac{4(x-1)(x+1)}{3x^2}\right]$$

$$y = \frac{1}{2}\log\left[\frac{4(x^2-1)}{3x^2}\right]$$

Question 13:

$$\cos\left(\frac{dy}{dx}\right) = a\,(a \in R);\ y = 1 \text{ when } x = 0$$

Answer

$$\cos\left(\frac{dy}{dx}\right)=a$$
$$\Rightarrow \frac{dy}{dx}=\cos^{-1}a$$
$$\Rightarrow dy=\cos^{-1}a\,dx$$

Integrating both sides, we get:

$$\int dy=\cos^{-1}a\int dx$$
$$\Rightarrow y=\cos^{-1}a\cdot x+C$$
$$\Rightarrow y=x\cos^{-1}a+C \qquad ...(1)$$

Now, $y=1$ when $x=0$.

$$\Rightarrow 1=0\cdot\cos^{-1}a+C$$
$$\Rightarrow C=1$$

Substituting C = 1 in equation (1), we get:

$$y=x\cos^{-1}a+1$$
$$\Rightarrow \frac{y-1}{x}=\cos^{-1}a$$
$$\Rightarrow \cos\left(\frac{y-1}{x}\right)=a$$

Question 14:

$$\frac{dy}{dx}=y\tan x;\ y=1 \text{ when } x=0$$

Answer

$$\frac{dy}{dx}=y\tan x$$
$$\Rightarrow \frac{dy}{y}=\tan x\,dx$$

Integrating both sides, we get:

$$\int \frac{dy}{y} = -\int \tan x\, dx$$

$$\Rightarrow \log y = \log(\sec x) + \log C$$

$$\Rightarrow \log y = \log(C \sec x)$$

$$\Rightarrow y = C \sec x \qquad ...(1)$$

Now, $y = 1$ when $x = 0$.

$$\Rightarrow 1 = C \times \sec 0$$

$$\Rightarrow 1 = C \times 1$$

$$\Rightarrow C = 1$$

Substituting C = 1 in equation (1), we get:

y = sec x

Question 15:

Find the equation of a curve passing through the point (0, 0) and whose differential equation is $y' = e^x \sin x$.

Answer

The differential equation of the curve is:

$$y' = e^x \sin x$$

$$\Rightarrow \frac{dy}{dx} = e^x \sin x$$

$$\Rightarrow dy = e^x \sin x$$

Integrating both sides, we get:

$$\int dy = \int e^x \sin x\, dx \qquad ...(1)$$

Let $I = \int e^x \sin x\, dx$.

$$\Rightarrow I = \sin x \int e^x dx - \int \left(\frac{d}{dx}(\sin x) \cdot \int e^x dx \right) dx$$

$$\Rightarrow I = \sin x \cdot e^x - \int \cos x \cdot e^x dx$$

$$\Rightarrow I = \sin x \cdot e^x - \left[\cos x \cdot \int e^x dx - \int \left(\frac{d}{dx}(\cos x) \cdot \int e^x dx \right) dx \right]$$

$$\Rightarrow I = \sin x \cdot e^x - \left[\cos x \cdot e^x - \int (-\sin x) \cdot e^x dx \right]$$

$$\Rightarrow I = e^x \sin x - e^x \cos x - I$$

$$\Rightarrow 2I = e^x (\sin x - \cos x)$$

$$\Rightarrow I = \frac{e^x (\sin x - \cos x)}{2}$$

Substituting this value in equation (1), we get:

$$y = \frac{e^x (\sin x - \cos x)}{2} + C \qquad ...(2)$$

Now, the curve passes through point (0, 0).

$$\therefore 0 = \frac{e^0 (\sin 0 - \cos 0)}{2} + C$$

$$\Rightarrow 0 = \frac{1(0-1)}{2} + C$$

$$\Rightarrow C = \frac{1}{2}$$

Substituting $C = \frac{1}{2}$ in equation (2), we get:

$$y = \frac{e^x (\sin x - \cos x)}{2} + \frac{1}{2}$$

$$\Rightarrow 2y = e^x (\sin x - \cos x) + 1$$

$$\Rightarrow 2y - 1 = e^x (\sin x - \cos x)$$

Hence, the required equation of the curve is $2y - 1 = e^x (\sin x - \cos x)$.

Question 16:

For the differential equation $xy\frac{dy}{dx}=(x+2)(y+2),$ find the solution curve passing through the point (1, −1).

Answer

The differential equation of the given curve is:

$$xy\frac{dy}{dx}=(x+2)(y+2)$$

$$\Rightarrow\left(\frac{y}{y+2}\right)dy=\left(\frac{x+2}{x}\right)dx$$

$$\Rightarrow\left(1-\frac{2}{y+2}\right)dy=\left(1+\frac{2}{x}\right)dx$$

Integrating both sides, we get:

$$\int\left(1-\frac{2}{y+2}\right)dy=\int\left(1+\frac{2}{x}\right)dx$$

$$\Rightarrow\int dy-2\int\frac{1}{y+2}dy=\int dx+2\int\frac{1}{x}dx$$

$$\Rightarrow y-2\log(y+2)=x+2\log x+C$$

$$\Rightarrow y-x-C=\log x^2+\log(y+2)^2$$

$$\Rightarrow y-x-C=\log\left[x^2(y+2)^2\right] \qquad ...(1)$$

Now, the curve passes through point (1, −1).

$$\Rightarrow -1-1-C=\log\left[(1)^2(-1+2)^2\right]$$

$$\Rightarrow -2-C=\log 1=0$$

$$\Rightarrow C=-2$$

Substituting C = −2 in equation (1), we get:

$$y-x+2=\log\left[x^2(y+2)^2\right]$$

This is the required solution of the given curve.

Question 17:

Find the equation of a curve passing through the point (0, –2) given that at any point (x, y) on the curve, the product of the slope of its tangent and y-coordinate of the point is equal to the x-coordinate of the point.

Answer

Let x and y be the x-coordinate and y-coordinate of the curve respectively.

We know that the slope of a tangent to the curve in the coordinate axis is given by the relation,

$$\frac{dy}{dx}$$

According to the given information, we get:

$$y \cdot \frac{dy}{dx} = x$$

$$\Rightarrow y\,dy = x\,dx$$

Integrating both sides, we get:

$$\int y\,dy = \int x\,dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + \mathrm{C}$$

$$\Rightarrow y^2 - x^2 = 2\mathrm{C} \qquad \ldots(1)$$

Now, the curve passes through point (0, –2).

$\therefore (-2)^2 - 0^2 = 2C$

$\Rightarrow 2C = 4$

Substituting 2C = 4 in equation (1), we get:

$y^2 - x^2 = 4$

This is the required equation of the curve.

Question 18:

At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (–4, –3). Find the equation of the curve given that it passes through (–2, 1).

Answer

It is given that (x, y) is the point of contact of the curve and its tangent.

The slope (m_1) of the line segment joining (x, y) and (−4, −3) is $\frac{y+3}{x+4}$.

We know that the slope of the tangent to the curve is given by the relation,

$$\therefore \text{Slope } (m_2) \text{ of the tangent } = \frac{dy}{dx}$$

According to the given information:

$$m_2 = 2m_1$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(y+3)}{x+4}$$

$$\Rightarrow \frac{dy}{y+3} = \frac{2dx}{x+4}$$

Integrating both sides, we get:

$$\int \frac{dy}{y+3} = 2\int \frac{dx}{x+4}$$

$$\Rightarrow \log(y+3) = 2\log(x+4) + \log C$$

$$\Rightarrow \log(y+3) \log C(x+4)^2$$

$$\Rightarrow y+3 = C(x+4)^2 \qquad \ldots(1)$$

This is the general equation of the curve.

It is given that it passes through point (−2, 1).

$$\Rightarrow 1+3 = C(-2+4)^2$$

$$\Rightarrow 4 = 4C$$

$$\Rightarrow C = 1$$

Substituting C = 1 in equation (1), we get:

$y + 3 = (x + 4)^2$

This is the required equation of the curve.

Question 19:

The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds.

Answer

Let the rate of change of the volume of the balloon be k (where k is a constant).

$$\Rightarrow \frac{dv}{dt} = k$$

$$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k \qquad \left[\text{Volume of sphere} = \frac{4}{3}\pi r^3\right]$$

$$\Rightarrow \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} = k$$

$$\Rightarrow 4\pi r^2\, dr = k\, dt$$

Integrating both sides, we get:

$$4\pi \int r^2 dr = k \int dt$$

$$\Rightarrow 4\pi \cdot \frac{r^3}{3} = kt + C$$

$$\Rightarrow 4\pi r^3 = 3(kt + C) \qquad \ldots(1)$$

Now, at $t = 0$, $r = 3$:

$\Rightarrow 4\pi \times 3^3 = 3(k \times 0 + C)$

$\Rightarrow 108\pi = 3C$

$\Rightarrow C = 36\pi$

At $t = 3$, $r = 6$:

$\Rightarrow 4\pi \times 6^3 = 3(k \times 3 + C)$

$\Rightarrow 864\pi = 3(3k + 36\pi)$

$\Rightarrow 3k = -288\pi - 36\pi = 252\pi$

$\Rightarrow k = 84\pi$

Substituting the values of k and C in equation (1), we get:

$$4\pi r^3 = 3[84\pi t + 36\pi]$$
$$\Rightarrow 4\pi r^3 = 4\pi(63t + 27)$$
$$\Rightarrow r^3 = 63t + 27$$
$$\Rightarrow r = (63t + 27)^{\frac{1}{3}}$$

Thus, the radius of the balloon after t seconds is $(63t + 27)^{\frac{1}{3}}$.

Question 20:

In a bank, principal increases continuously at the rate of r% per year. Find the value of r if Rs 100 doubles itself in 10 years ($\log_e 2 = 0.6931$).

Answer

Let p, t, and r represent the principal, time, and rate of interest respectively.

It is given that the principal increases continuously at the rate of r% per year.

$$\Rightarrow \frac{dp}{dt} = \left(\frac{r}{100}\right)p$$
$$\Rightarrow \frac{dp}{p} = \left(\frac{r}{100}\right)dt$$

Integrating both sides, we get:

$$\int \frac{dp}{p} = \frac{r}{100}\int dt$$
$$\Rightarrow \log p = \frac{rt}{100} + k$$
$$\Rightarrow p = e^{\frac{rt}{100}+k} \qquad \ldots(1)$$

It is given that when t = 0, p = 100.

⇒ 100 = e^k … (2)

Now, if t = 10, then p = 2 × 100 = 200.

Therefore, equation (1) becomes:

$$200 = e^{\frac{r}{10}+k}$$

$$\Rightarrow 200 = e^{\frac{r}{10}} \cdot e^{k}$$

$$\Rightarrow 200 = e^{\frac{r}{10}} \cdot 100 \qquad \text{(From (2))}$$

$$\Rightarrow e^{\frac{r}{10}} = 2$$

$$\Rightarrow \frac{r}{10} = \log_e 2$$

$$\Rightarrow \frac{r}{10} = 0.6931$$

$$\Rightarrow r = 6.931$$

Hence, the value of r is 6.93%.

Question 21:

In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years $\left(e^{0.5} = 1.648\right)$.

Answer

Let p and t be the principal and time respectively.

It is given that the principal increases continuously at the rate of 5% per year.

$$\Rightarrow \frac{dp}{dt} = \left(\frac{5}{100}\right)p$$

$$\Rightarrow \frac{dp}{dt} = \frac{p}{20}$$

$$\Rightarrow \frac{dp}{p} = \frac{dt}{20}$$

Integrating both sides, we get:

$$\int \frac{dp}{p} = \frac{1}{20}\int dt$$

$$\Rightarrow \log p = \frac{t}{20} + C$$

$$\Rightarrow p = e^{\frac{t}{20}+C} \qquad \ldots(1)$$

Now, when t = 0, p = 1000.

⇒ $1000 = e^C$ … (2)

At t = 10, equation (1) becomes:

$$p = e^{\frac{1}{2}+C}$$

$$\Rightarrow p = e^{0.5} \times e^C$$

$$\Rightarrow p = 1.648 \times 1000$$

$$\Rightarrow p = 1648$$

Hence, after 10 years the amount will worth Rs 1648.

Question 22:

In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

Answer

Let y be the number of bacteria at any instant t.

It is given that the rate of growth of the bacteria is proportional to the number present.

$$\therefore \frac{dy}{dt} \propto y$$

$$\Rightarrow \frac{dy}{dt} = ky \text{ (where } k \text{ is a constant)}$$

$$\Rightarrow \frac{dy}{y} = k dt$$

Integrating both sides, we get:

$$\int \frac{dy}{y} = k \int dt$$

$$\Rightarrow \log y = kt + C \qquad \text{...(1)}$$

Let y_0 be the number of bacteria at t = 0.

⇒ $\log y_0 = C$

Substituting the value of C in equation (1), we get:

$$\log y = kt + \log y_0$$
$$\Rightarrow \log y - \log y_0 = kt$$
$$\Rightarrow \log\left(\frac{y}{y_0}\right) = kt$$
$$\Rightarrow kt = \log\left(\frac{y}{y_0}\right) \qquad ...(2)$$

Also, it is given that the number of bacteria increases by 10% in 2 hours.

$$\Rightarrow y = \frac{110}{100} y_0$$
$$\Rightarrow \frac{y}{y_0} = \frac{11}{10} \qquad ...(3)$$

Substituting this value in equation (2), we get:

$$k \cdot 2 = \log\left(\frac{11}{10}\right)$$
$$\Rightarrow k = \frac{1}{2}\log\left(\frac{11}{10}\right)$$

Therefore, equation (2) becomes:

$$\frac{1}{2}\log\left(\frac{11}{10}\right) \cdot t = \log\left(\frac{y}{y_0}\right)$$
$$\Rightarrow t = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} \qquad ...(4)$$

Now, let the time when the number of bacteria increases from 100000 to 200000 be t_1.

$\Rightarrow y = 2y_0$ at $t = t_1$

From equation (4), we get:

$$t_1 = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} = \frac{2\log 2}{\log\left(\frac{11}{10}\right)}$$

Hence, in $\frac{2\log 2}{\log\left(\frac{11}{10}\right)}$ hours the number of bacteria increases from 100000 to 200000.

Question 23:

The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

A. $e^x + e^{-y} = C$

B. $e^x + e^y = C$

C. $e^{-x} + e^y = C$

D. $e^{-x} + e^{-y} = C$

Answer

$$\frac{dy}{dx} = e^{x+y} = e^x \cdot e^y$$

$$\Rightarrow \frac{dy}{e^y} = e^x dx$$

$$\Rightarrow e^{-y} dy = e^x dx$$

Integrating both sides, we get:

$$\int e^{-y} dy = \int e^x dx$$

$$\Rightarrow -e^{-y} = e^x + k$$

$$\Rightarrow e^x + e^{-y} = -k$$

$$\Rightarrow e^x + e^{-y} = c \qquad (c = -k)$$

Hence, the correct answer is A.