Exercise 9.5

Question 1:

$\left(x^2+xy\right)dy=\left(x^2+y^2\right)dx$

Answer

The given differential equation i.e., $(x^2 + xy)\ dy = (x^2 + y^2)\ dx$ can be written as:

$$\frac{dy}{dx}=\frac{x^2+y^2}{x^2+xy} \qquad \ldots(1)$$

Let $F\left(x,y\right)=\dfrac{x^2+y^2}{x^2+xy}$.

Now, $F\left(\lambda x,\lambda y\right)=\dfrac{\left(\lambda x\right)^2+\left(\lambda y\right)^2}{\left(\lambda x\right)^2+\left(\lambda x\right)\left(\lambda y\right)}=\dfrac{x^2+y^2}{x^2+xy}=\lambda^0\cdot F\left(x,y\right)$

This shows that equation (1) is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of v and $\dfrac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{x^2 + (vx)^2}{x^2 + x(vx)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^2}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v = \frac{(1+v^2) - v(1+v)}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v}$$

$$\Rightarrow \left(\frac{1+v}{1-v}\right) = dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{2-1+v}{1-v}\right)dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{2}{1-v} - 1\right)dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$-2\log(1-v) - v = \log x - \log k$$

$$\Rightarrow v = -2\log(1-v) - \log x + \log k$$

$$\Rightarrow v = \log\left[\frac{k}{x(1-v)^2}\right]$$

$$\Rightarrow \frac{y}{x} = \log\left[\frac{k}{x\left(1-\frac{y}{x}\right)^2}\right]$$

$$\Rightarrow \frac{y}{x} = \log\left[\frac{kx}{(x-y)^2}\right]$$

$$\Rightarrow \frac{kx}{(x-y)^2} = e^{\frac{y}{x}}$$

$$\Rightarrow (x-y)^2 = kxe^{-\frac{y}{x}}$$

This is the required solution of the given differential equation.

Question 2:

$$y' = \frac{x+y}{x}$$

Answer

The given differential equation is:

$$y' = \frac{x+y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x} \qquad \ldots(1)$$

$$\text{Let } F(x, y) = \frac{x+y}{x}.$$

$$\text{Now, } F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x} = \frac{x+y}{x} = \lambda^0 F(x, y)$$

Thus, the given equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{x+vx}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = 1 + v$$

$$x\frac{dv}{dx} = 1$$

$$\Rightarrow dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$v = \log x + C$$

$$\Rightarrow \frac{y}{x} = \log x + C$$

$$\Rightarrow y = x \log x + Cx$$

This is the required solution of the given differential equation.

Question 3:

$$(x-y)dy-(x+y)dx=0$$

Answer

The given differential equation is:

$$(x-y)dy-(x+y)dx=0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y} \qquad ...(1)$$

$$\text{Let } F(x,y) = \frac{x+y}{x-y}.$$

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x - \lambda y} = \frac{x+y}{x-y} = \lambda^0 \cdot F(x,y)$$

Thus, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$

$$v+x\frac{dv}{dx}=\frac{x+vx}{x-vx}=\frac{1+v}{1-v}$$

$$x\frac{dv}{dx}=\frac{1+v}{1-v}-v=\frac{1+v-v(1-v)}{1-v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v^2}{1-v}$$

$$\Rightarrow \frac{1-v}{\left(1+v^2\right)}dv=\frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{1+v^2}-\frac{v}{1-v^2}\right)dv=\frac{dx}{x}$$

Integrating both sides, we get:

$$\tan^{-1}v-\frac{1}{2}\log\left(1+v^2\right)=\log x+\mathrm{C}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\log\left[1+\left(\frac{y}{x}\right)^2\right]=\log x+\mathrm{C}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\log\left(\frac{x^2+y^2}{x^2}\right)=\log x+\mathrm{C}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\left[\log\left(x^2+y^2\right)-\log x^2\right]=\log x+\mathrm{C}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\log\left(x^2+y^2\right)+\mathrm{C}$$

This is the required solution of the given differential equation.

Question 4:

$$\left(x^2-y^2\right)dx+2xy\ dy=0$$

Answer

The given differential equation is:

$$\left(x^2-y^2\right)dx+2xy\ dy=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-\left(x^2-y^2\right)}{2xy} \qquad \ldots(1)$$

Let $F(x,y)=\dfrac{-\left(x^2-y^2\right)}{2xy}$.

$$\therefore\ F(\lambda x,\lambda y)=\left[\frac{(\lambda x)^2-(\lambda y)^2}{2(\lambda x)(\lambda y)}\right]=\frac{-\left(x^2-y^2\right)}{2xy}=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\dfrac{dy}{dx}$

$$v+x\frac{dv}{dx}=-\left[\frac{x^2-(vx)^2}{2x\cdot(vx)}\right]$$

$$v+x\frac{dv}{dx}=\frac{v^2-1}{2v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2-1}{2v}-v=\frac{v^2-1-2v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx}=-\frac{\left(1+v^2\right)}{2v}$$

$$\Rightarrow \frac{2v}{1+v^2}dv=-\frac{dx}{x}$$

Integrating both sides, we get:

$$\log\left(1+v^2\right)=-\log x+\log C=\log\frac{C}{x}$$

$$\Rightarrow 1+v^2=\frac{C}{x}$$

$$\Rightarrow\left[1+\frac{y^2}{x^2}\right]=\frac{C}{x}$$

$$\Rightarrow x^2+y^2=Cx$$

This is the required solution of the given differential equation.

Question 5:

$$x^2\frac{dy}{dx}-x^2-2y^2+xy$$

Answer

The given differential equation is:

$$x^2\frac{dy}{dx}=x^2-2y^2+xy$$

$$\frac{dy}{dx}=\frac{x^2-2y^2+xy}{x^2} \qquad \ldots(1)$$

$$\text{Let } F(x,y)=\frac{x^2-2y^2+xy}{x^2}.$$

$$\therefore\ F(\lambda x,\lambda y)=\frac{(\lambda x)^2-2(\lambda y)^2+(\lambda x)(\lambda y)}{(\lambda x)^2}=\frac{x^2-2y^2+xy}{x^2}=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

$$\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$

$$v+x\frac{dv}{dx}=\frac{x^2-2(vx)^2+x\cdot(vx)}{x^2}$$

$$\Rightarrow v+x\frac{dv}{dx}=1-2v^2+v$$

$$\Rightarrow x\frac{dv}{dx}=1-2v^2$$

$$\Rightarrow \frac{dv}{1-2v^2}=\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\cdot\frac{dv}{\frac{1}{2}-v^2}=\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\cdot\left[\frac{dv}{\left(\frac{1}{\sqrt{2}}\right)^2-v^2}\right]=\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\cdot\frac{1}{2\times\frac{1}{\sqrt{2}}}\log\left|\frac{\frac{1}{\sqrt{2}}+v}{\frac{1}{\sqrt{2}}-v}\right|=\log|x|+C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{\frac{1}{\sqrt{2}}+\frac{y}{x}}{\frac{1}{\sqrt{2}}-\frac{y}{x}}\right|=\log|x|+C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{x+\sqrt{2}y}{x-\sqrt{2}y}\right|=\log|x|+C$$

This is the required solution for the given differential equation.

Question 6:

$xdy - ydx = \sqrt{x^2 + y^2}\,dx$

Answer

$$xdy - ydx = \sqrt{x^2 + y^2}\,dx$$

$$\Rightarrow xdy = \left[y + \sqrt{x^2 + y^2}\right]dx$$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x^2} \qquad ...(1)$$

$$\text{Let } F(x, y) = \frac{y + \sqrt{x^2 + y^2}}{x^2}.$$

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda x + \sqrt{(\lambda x)^2 + (\lambda y)^2}}{\lambda x} = \frac{y + \sqrt{x^2 + y^2}}{x} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of v and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + (vx)^2}}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v + \sqrt{1 + v^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

Integrating both sides, we get:

$$\log\left|v+\sqrt{1+v^2}\right| = \log|x| + \log C$$

$$\Rightarrow \log\left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right| = \log|Cx|$$

$$\Rightarrow \log\left|\frac{y+\sqrt{x^2+y^2}}{x}\right| = \log|Cx|$$

$$\Rightarrow y+\sqrt{x^2+y^2} = Cx^2$$

This is the required solution of the given differential equation.

Question 7:

$$\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx = \left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdy$$

Answer

The given differential equation is:

$$\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx = \left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdy$$

$$\frac{dy}{dx} = \frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x} \qquad ...(1)$$

$$\text{Let } F(x,y) = \frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}.$$

$$\therefore F(\lambda x, \lambda y) = \frac{\left\{\lambda x \cos\left(\frac{\lambda y}{\lambda x}\right) + \lambda y \sin\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda y}{\left\{\lambda y \sin\left(\frac{\lambda y}{\lambda x}\right) - \lambda x \sin\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda x}$$

$$= \frac{\left\{x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right)\right\}y}{\left\{y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right)\right\}x}$$

$$= \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

$$\Rightarrow \frac{dy}{dx} = v + x = \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v+x\frac{dv}{dx}=\frac{(x\cos v+vx\sin v)\cdot vx}{(vx\sin v-x\cos v)\cdot x}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v}{v\sin v-\cos v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v}{v\sin v-\cos v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v-v^2\sin v+v\cos v}{v\sin v-\cos v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{2v\cos v}{v\sin v-\cos v}$$

$$\Rightarrow\left[\frac{v\sin v-\cos v}{v\cos v}\right]dv=\frac{2dx}{x}$$

$$\Rightarrow\left(\tan v-\frac{1}{v}\right)dv=\frac{2dx}{x}$$

Integrating both sides, we get:

$$\log(\sec v)-\log v=2\log x+\log C$$

$$\Rightarrow\log\left(\frac{\sec v}{v}\right)=\log\left(Cx^2\right)$$

$$\Rightarrow\left(\frac{\sec v}{v}\right)=Cx^2$$

$$\Rightarrow\sec v=Cx^2v$$

$$\Rightarrow\sec\left(\frac{y}{x}\right)=C\cdot x^2\cdot\frac{y}{x}$$

$$\Rightarrow\sec\left(\frac{y}{x}\right)=Cxy$$

$$\Rightarrow\cos\left(\frac{y}{x}\right)=\frac{1}{Cxy}=\frac{1}{C}\cdot\frac{1}{xy}$$

$$\Rightarrow xy\cos\left(\frac{y}{x}\right)=k \qquad \left(k=\frac{1}{C}\right)$$

This is the required solution of the given differential equation.

Question 8:

$$x\frac{dy}{dx}-y+x\sin\left(\frac{y}{x}\right)=0$$

Answer

$$x\frac{dy}{dx}-y+x\sin\left(\frac{y}{x}\right)=0$$

$$\Rightarrow x\frac{dy}{dx}=y-x\sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{y-x\sin\left(\frac{y}{x}\right)}{x} \qquad ...(1)$$

$$\text{Let } F(x,y)=\frac{y-x\sin\left(\frac{y}{x}\right)}{x}.$$

$$\therefore\ F(\lambda x,\lambda y)=\frac{\lambda y-\lambda x\sin\left(\frac{\lambda y}{\lambda x}\right)}{\lambda x}=\frac{y-x\sin\left(\frac{y}{x}\right)}{x}=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{vx - x\sin v}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v - \sin v$$

$$\Rightarrow -\frac{dv}{\sin v} = \frac{dx}{x}$$

$$\Rightarrow \operatorname{cosec} v\, dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\log\left|\operatorname{cosec} v - \cot v\right| = -\log x + \log C = \log\frac{C}{x}$$

$$\Rightarrow \operatorname{cosec}\left(\frac{y}{x}\right) - \cot\left(\frac{y}{x}\right) = \frac{C}{x}$$

$$\Rightarrow \frac{1}{\sin\left(\frac{y}{x}\right)} - \frac{\cos\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)} = \frac{C}{x}$$

$$\Rightarrow x\left[1 - \cos\left(\frac{y}{x}\right)\right] = C\sin\left(\frac{y}{x}\right)$$

This is the required solution of the given differential equation.

Question 9:

$$ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$$

Answer

$$ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$$

$$\Rightarrow ydx = \left[2x - x\log\left(\frac{y}{x}\right)\right]dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} \qquad ...(1)$$

$$\text{Let } F(x, y) = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}.$$

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda y}{2(\lambda x) - (\lambda x)\log\left(\frac{\lambda y}{\lambda x}\right)} = \frac{y}{2x - \log\left(\frac{y}{x}\right)} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$

$$v+x\frac{dv}{dx}=\frac{vx}{2x-x\log v}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v}{2-\log v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v}{2-\log v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v-2v+v\log v}{2-\log v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v\log v-v}{2-\log v}$$

$$\Rightarrow \frac{2-\log v}{v(\log v-1)}dv=\frac{dx}{x}$$

$$\Rightarrow \left[\frac{1+(1-\log v)}{v(\log v-1)}\right]dv=\frac{dx}{x}$$

$$\Rightarrow \left[\frac{1}{v(\log v-1)}-\frac{1}{v}\right]dv=\frac{dx}{x}$$

Integrating both sides, we get:

$$\int\frac{1}{v(\log v-1)}dv-\int\frac{1}{v}dv=\int\frac{1}{x}dx$$

$$\Rightarrow \int\frac{dv}{v(\log v-1)}-\log v=\log x+\log C \qquad ...(2)$$

$$\Rightarrow \text{Let } \log v-1=t$$

$$\Rightarrow \frac{d}{dv}(\log v-1)=\frac{dt}{dv}$$

$$\Rightarrow \frac{1}{v}=\frac{dt}{dv}$$

$$\Rightarrow \frac{dv}{v}=dt$$

Therefore, equation (1) becomes:

$$\Rightarrow \int \frac{dt}{t} - \log v = \log x + \log C$$

$$\Rightarrow \log t - \log\left(\frac{y}{x}\right) = \log(Cx)$$

$$\Rightarrow \log\left[\log\left(\frac{y}{x}\right) - 1\right] - \log\left(\frac{y}{x}\right) = \log(Cx)$$

$$\Rightarrow \log\left[\frac{\log\left(\frac{y}{x}\right) - 1}{\frac{y}{x}}\right] = \log(Cx)$$

$$\Rightarrow \frac{x}{y}\left[\log\left(\frac{y}{x}\right) - 1\right] = Cx$$

$$\Rightarrow \log\left(\frac{y}{x}\right) - 1 = Cy$$

This is the required solution of the given differential equation.

Question 10:

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

Answer

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

$$\Rightarrow \left(1 + e^{\frac{x}{y}}\right)dx = -e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}} \qquad ...(1)$$

$$\text{Let } F(x,y) = \frac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}}.$$

$$\therefore\ F(\lambda x, \lambda y) = \frac{-e^{\frac{\lambda x}{\lambda y}}\left(1-\frac{\lambda x}{\lambda y}\right)}{1+e^{\frac{\lambda x}{\lambda y}}} = \frac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}} = \lambda^0 \cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

x = vy

$$\Rightarrow \frac{d}{dy}(x) = \frac{d}{dy}(vy)$$

$$\Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the values of x and $\frac{dx}{dy}$

$$v + y\frac{dv}{dy} = \frac{-e^v(1-v)}{1+e^v}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^v + ve^v}{1+e^v} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^v + ve^v - v - ve^v}{1+e^v}$$

$$\Rightarrow y\frac{dv}{dy} = -\left[\frac{v+e^v}{1+e^v}\right]$$

$$\Rightarrow \left[\frac{1+e^v}{v+e^v}\right]dv = -\frac{dy}{y}$$

Integrating both sides, we get:

$$\Rightarrow \log\left(v+e^{v}\right)=-\log y+\log C=\log\left(\frac{C}{y}\right)$$

$$\Rightarrow \left[\frac{x}{y}+e^{\frac{x}{y}}\right]=\frac{C}{y}$$

$$\Rightarrow x+ye^{\frac{x}{y}}=C$$

This is the required solution of the given differential equation.

Question 11:

$$(x+y)dy+(x-y)dy=0; y=1 \text{ when } x=1$$

Answer

$$(x+y)dy+(x-y)dx=0$$

$$\Rightarrow (x+y)dy=-(x-y)dx$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-y)}{x+y} \qquad \ldots(1)$$

$$\text{Let } F(x,y)=\frac{-(x-y)}{x+y}.$$

$$\therefore F(\lambda x,\lambda y)=\frac{-(\lambda x-\lambda y)}{\lambda x-\lambda y}=\frac{-(x-y)}{x+y}=\lambda^{0}\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v+x\frac{dv}{dx}=\frac{-(x-vx)}{x+vx}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v-1}{v+1}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v-1}{v+1}-v=\frac{v-1-v(v+1)}{v+1}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v-1-v^2-v}{v+1}=\frac{-(1+v^2)}{v+1}$$

$$\Rightarrow \frac{(v+1)}{1+v^2}dv=-\frac{dx}{x}$$

$$\Rightarrow \left[\frac{v}{1+v^2}+\frac{1}{1+v^2}\right]dv=-\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\log(1+v^2)+\tan^{-1}v=-\log x+k$$

$$\Rightarrow \log(1+v^2)+2\tan^{-1}v=-2\log x+2k$$

$$\Rightarrow \log\left[(1+v^2)\cdot x^2\right]+2\tan^{-1}v=2k$$

$$\Rightarrow \log\left[\left(1+\frac{y^2}{x^2}\right)\cdot x^2\right]+2\tan^{-1}\frac{y}{x}=2k$$

$$\Rightarrow \log(x^2+y^2)+2\tan^{-1}\frac{y}{x}=2k \qquad ...(2)$$

Now, y = 1 at x = 1.

$$\Rightarrow \log 2+2\tan^{-1}1=2k$$

$$\Rightarrow \log 2+2\times\frac{\pi}{4}=2k$$

$$\Rightarrow \frac{\pi}{2}+\log 2=2k$$

Substituting the value of 2k in equation (2), we get:

$$\log(x^2+y^2)+2\tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{2}+\log 2$$

This is the required solution of the given differential equation.

Question 12:

$x^2dy + (xy + y^2)dx = 0; y = 1 \text{ when } x = 1$

Answer

$x^2dy + (xy + y^2)dx = 0$

$\Rightarrow x^2dy = -(xy + y^2)dx$

$\Rightarrow \frac{dy}{dx} = \frac{-(xy + y^2)}{x^2} \qquad ...(1)$

Let $F(x, y) = \frac{-(xy + y^2)}{x^2}$.

$\therefore\ F(\lambda x, \lambda y) = \frac{[\lambda x \cdot \lambda y + (\lambda y)^2]}{(\lambda x)^2} = \frac{-(xy + y^2)}{x^2} = \lambda^0 \cdot F(x, y)$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting the values of y and $\frac{dy}{dx}$

$$v+x\frac{dv}{dx}=\frac{-\left[x\cdot vx+(vx)^2\right]}{x^2}=-v-v^2$$

$$\Rightarrow x\frac{dv}{dx}=-v^2-2v=-v(v+2)$$

$$\Rightarrow \frac{dv}{v(v+2)}=-\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left[\frac{(v+2)-v}{v(v+2)}\right]dv=-\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{v}-\frac{1}{v+2}\right]dv=-\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\left[\log v-\log(v+2)\right]=-\log x+\log C$$

$$\Rightarrow \frac{1}{2}\log\left(\frac{v}{v+2}\right)=\log\frac{C}{x}$$

$$\Rightarrow \frac{v}{v+2}=\left(\frac{C}{x}\right)^2$$

$$\Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+2}=\left(\frac{C}{x}\right)^2$$

$$\Rightarrow \frac{y}{y+2x}=\frac{C^2}{x^2}$$

$$\Rightarrow \frac{x^2y}{y+2x}=C^2 \qquad \ldots(2)$$

Now, y = 1 at x = 1.

$$\Rightarrow \frac{1}{1+2} = C^2$$

$$\Rightarrow C^2 = \frac{1}{3}$$

Substituting $C^2 = \frac{1}{3}$

$$\frac{x^2 y}{y+2x} = \frac{1}{3}$$

$$\Rightarrow y + 2x = 3x^2 y$$

This is the required solution of the given differential equation.

Question 13:

$$\left[x\sin^2\left(\frac{x}{y} - y\right)\right]dx + xdy = 0; y\frac{\pi}{4} \text{ when } x = 1$$

Answer

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + xdy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x} \qquad \ldots(1)$$

$$\text{Let } F(x, y) = \frac{-\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x}.$$

$$\therefore\ F(\lambda x, \lambda y) = \frac{-\left[\lambda x \cdot \sin^2\left(\frac{\lambda x}{\lambda y}\right) - \lambda y\right]}{\lambda x} = \frac{-\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve this differential equation, we make the substitution as:

y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$
$$\Rightarrow \frac{dy}{dx} = v + x = \frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{-\left[x\sin^2 v - vx\right]}{x}$$
$$\Rightarrow v + x\frac{dv}{dx} = -\left[\sin^2 v - v\right] = v - \sin^2 v$$
$$\Rightarrow x\frac{dv}{dx} = -\sin^2 v$$
$$\Rightarrow \frac{dv}{\sin^2 v} = -\frac{dx}{dx}$$
$$\Rightarrow \text{cosec}^2 v dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$-\cot v = -\log|x| - C$$
$$\Rightarrow \cot v = \log|x| + C$$
$$\Rightarrow \cot\left(\frac{y}{x}\right) = \log|x| + \log C$$
$$\Rightarrow \cot\left(\frac{y}{x}\right) = \log|Cx| \qquad ...(2)$$

Now, $y = \frac{\pi}{4}$ at $x = 1$.

$$\Rightarrow \cot\left(\frac{\pi}{4}\right) = \log|C|$$
$$\Rightarrow 1 = \log C$$
$$\Rightarrow C = e^1 = e$$

Substituting C = e in equation (2), we get:

$$\cot\left(\frac{y}{x}\right)=\log|ex|$$

This is the required solution of the given differential equation.

Question 14:

$$\frac{dy}{dx}-\frac{y}{x}+\operatorname{cosec}\left(\frac{y}{x}\right)=0;\ y=0 \text{ when } x=1$$

Answer

$$\frac{dy}{dx}-\frac{y}{x}+\operatorname{cosec}\left(\frac{y}{x}\right)=0$$

$$\Rightarrow\frac{dy}{dx}=\frac{y}{x}-\operatorname{cosec}\left(\frac{y}{x}\right) \qquad ...(1)$$

$$\text{Let } F(x,y)=\frac{y}{x}-\operatorname{cosec}\left(\frac{y}{x}\right).$$

$$\therefore\ F(\lambda x,\lambda y)=\frac{\lambda y}{\lambda x}-\operatorname{cosec}\left(\frac{\lambda y}{\lambda x}\right)$$

$$\Rightarrow\ F(\lambda x,\lambda y)=\frac{y}{x}-\operatorname{cosec}\left(\frac{y}{x}\right)=F(x,y)=\lambda^0\cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:

y = vx

$$\Rightarrow\frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = v - \operatorname{cosec} v$$
$$\Rightarrow -\frac{dv}{\operatorname{cosec} v} = -\frac{dx}{x}$$
$$\Rightarrow -\sin v dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$\cos v = \log x + \log C = \log|Cx|$$
$$\Rightarrow \cos\left(\frac{y}{x}\right) = \log|Cx| \qquad ...(2)$$

This is the required solution of the given differential equation.

Now, y = 0 at x = 1.

$$\Rightarrow \cos(0) = \log C$$
$$\Rightarrow 1 = \log C$$
$$\Rightarrow C = e^1 = e$$

Substituting C = e in equation (2), we get:

$$\cos\left(\frac{y}{x}\right) = \log|(ex)|$$

This is the required solution of the given differential equation.

Question 15:

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0;\ y = 2 \text{ when } x = 1$$

Answer

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0$$

$$\Rightarrow 2x^2\frac{dy}{dx} = 2xy + y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy + y^2}{2x^2} \qquad \ldots(1)$$

$$\text{Let } F(x, y) = \frac{2xy + y^2}{2x^2}.$$

$$\therefore\ F(\lambda x, \lambda y) = \frac{2(\lambda x)(\lambda y) + (\lambda y)^2}{2(\lambda x)^2} = \frac{2xy + y^2}{2x^2} = \lambda^0 \cdot F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{2x(vx) + (vx)^2}{2x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{2v + v^2}{2}$$

$$\Rightarrow v + x\frac{dv}{dx} = v + \frac{v^2}{2}$$

$$\Rightarrow \frac{2}{v^2}dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$2\cdot\frac{v^{-2+1}}{-2+1}=\log|x|+C$$
$$\Rightarrow -\frac{2}{v}=\log|x|+C$$
$$\Rightarrow -\frac{2}{\frac{y}{x}}=\log|x|+C$$
$$\Rightarrow -\frac{2x}{y}=\log|x|+C \qquad ...(2)$$

Now, y = 2 at x = 1.

$$\Rightarrow -1=\log(1)+C$$
$$\Rightarrow C=-1$$

Substituting C = −1 in equation (2), we get:

$$-\frac{2x}{y}=\log|x|-1$$
$$\Rightarrow \frac{2x}{y}=1-\log|x|$$
$$\Rightarrow y=\frac{2x}{1-\log|x|},\ (x\neq 0,\ x\neq e)$$

This is the required solution of the given differential equation.

Question 16:

A homogeneous differential equation of the form $\frac{dx}{dy}=h\left(\frac{x}{y}\right)$ can be solved by making the substitution

A. y = vx

B. v = yx

C. x = vy

D. x = v

Answer

For solving the homogeneous equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$, we need to make the substitution as x = vy.

Hence, the correct answer is C.

Question 17:

Which of the following is a homogeneous differential equation?

A. $(4x+6y+5)dy-(3y+2x+4)dx=0$

B. $(xy)dx-(x^3+y^3)dy=0$

C. $(x^3+2y^2)dx+2xy\,dy=0$

D. $y^2dx+(x^2-xy^2-y^2)dy=0$

Answer

Function F(x, y) is said to be the homogenous function of degree n, if

$F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any non-zero constant (λ).

Consider the equation given in alternative D:

$$y^2dx+\left(x^2-xy-y^2\right)dy=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-y^2}{x^2-xy-y^2}=\frac{y^2}{y^2+xy-x^2}$$

Let $F(x,y)=\frac{y^2}{y^2+xy-x^2}$.

$$\Rightarrow F(\lambda x,\lambda y)=\frac{(\lambda y)^2}{(\lambda y)^2+(\lambda x)(\lambda y)-(\lambda x)^2}$$

$$=\frac{\lambda^2y^2}{\lambda^2\left(y^2+xy-x^2\right)}$$

$$=\lambda^0\left(\frac{y^2}{y^2+xy-x^2}\right)$$

$$=\lambda^0\cdot F(x,y)$$

Hence, the differential equation given in alternative D is a homogenous equation.