Mathematics

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(Chapter 9)(Differential Equations)

XII

Miscellaneous Exercise 9

Question 1:

For each of the differential equations given below, indicate its order and degree (if defined).

(i) $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$

(ii) $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$

(iii) $\frac{d^4y}{dx^4}-\sin\left(\frac{d^3y}{dx^3}\right)=0$

Answer

(i) The differential equation is given as:

$$\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$$

$$\Rightarrow\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y-\log x=0$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$. Thus, its order is two. The highest power raised to $\frac{d^2y}{dx^2}$ is one. Hence, its degree is one.

(ii) The differential equation is given as:

$$\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$$

$$\Rightarrow\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y-\sin x=0$$

The highest order derivative present in the differential equation is $\frac{dy}{dx}$. Thus, its order is one. The highest power raised to $\frac{dy}{dx}$ is three. Hence, its degree is three.

(iii) The differential equation is given as:

$$\frac{d^4y}{dx^4} - \sin\left(\frac{d^3y}{dx^3}\right) = 0$$

The highest order derivative present in the differential equation is $\frac{d^4y}{dx^4}$. Thus, its order is four.

However, the given differential equation is not a polynomial equation. Hence, its degree is not defined.

Question 2:

For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

(i) $y = ae^x + be^{-x} + x^2$: $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$

(ii) $y = e^x(a\cos x + b\sin x)$: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

(iii) $y = x\sin 3x$: $\frac{d^2y}{dx^2} + 9y - 6\cos 3x = 0$

(iv) $x^2 = 2y^2\log y$: $(x^2 + y^2)\frac{dy}{dx} - xy = 0$

Answer

(i) $y = ae^x + be^{-x} + x^2$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = a\frac{d}{dx}\left(e^{x}\right) + b\frac{d}{dx}\left(e^{-x}\right) + \frac{d}{dx}\left(x^{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = ae^{x} - be^{-x} + 2x$$

Again, differentiating both sides with respect to x, we get:

$$\frac{d^2 y}{dx^2} = ae^{x} + be^{-x} + 2$$

Now, on substituting the values of $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ in the differential equation, we get:

L.H.S.

$$x\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2$$

$$= x\left(ae^{x} + be^{-x} + 2\right) + 2\left(ae^{x} - be^{-x} + 2x\right) - x\left(ae^{x} + be^{-x} + x^2\right) + x^2 - 2$$

$$= \left(axe^{x} + bxe^{-x} + 2x\right) + \left(2ae^{x} - 2be^{-x} + 4x\right) - \left(axe^{x} + bxe^{-x} + x^3\right) + x^2 - 2$$

$$= 2ae^{x} - 2be^{-x} + x^2 + 6x - 2$$

$$\neq 0$$

⇒ L.H.S. ≠ R.H.S.

Hence, the given function is not a solution of the corresponding differential equation.

(ii) $y = e^{x}\left(a\cos x + b\sin x\right) = ae^{x}\cos x + be^{x}\sin x$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = a\cdot\frac{d}{dx}\left(e^{x}\cos x\right) + b\cdot\frac{d}{dx}\left(e^{x}\sin x\right)$$

$$\Rightarrow \frac{dy}{dx} = a\left(e^{x}\cos x - e^{x}\sin x\right) + b\cdot\left(e^{x}\sin x + e^{x}\cos x\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(a+b\right)e^{x}\cos x + \left(b-a\right)e^{x}\sin x$$

Again, differentiating both sides with respect to x, we get:

$$\frac{d^2y}{dx^2}=(a+b)\cdot\frac{d}{dx}\left(e^x\cos x\right)+(b-a)\frac{d}{dx}\left(e^x\sin x\right)$$

$$\Rightarrow\frac{d^2y}{dx^2}=(a+b)\cdot\left[e^x\cos x-e^x\sin x\right]+(b-a)\left[e^x\sin x+e^x\cos x\right]$$

$$\Rightarrow\frac{d^2y}{dx^2}=e^x\left[(a+b)(\cos x-\sin x)+(b-a)(\sin x+\cos x)\right]$$

$$\Rightarrow\frac{d^2y}{dx^2}=e^x\left[a\cos x-a\sin x+b\cos x-b\sin x+b\sin x+b\cos x-a\sin x-a\cos x\right]$$

$$\Rightarrow\frac{d^2y}{dx^2}=\left[2e^x\left(b\cos x-a\sin x\right)\right]$$

Now, on substituting the values of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ in the L.H.S. of the given differential equation, we get:

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y$$
$$=2e^x\left(b\cos x-a\sin x\right)-2e^x\left[(a+b)\cos x+(b-a)\sin x\right]+2e^x\left(a\cos x+b\sin x\right)$$
$$=e^x\begin{bmatrix}(2b\cos x-2a\sin x)-(2a\cos x+2b\cos x)\\-(2b\sin x-2a\sin x)+(2a\cos x+2b\sin x)\end{bmatrix}$$
$$=e^x\left[(2b-2a-2b+2a)\cos x\right]+e^x\left[(-2a-2b+2a+2b)\sin x\right]$$
$$=0$$

Hence, the given function is a solution of the corresponding differential equation.

(iii) $y=x\sin 3x$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx}=\frac{d}{dx}\left(x\sin 3x\right)=\sin 3x+x\cdot\cos 3x\cdot 3$$

$$\Rightarrow\frac{dy}{dx}=\sin 3x+3x\cos 3x$$

Again, differentiating both sides with respect to x, we get:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(\sin 3x)+3\frac{d}{dx}(x\cos 3x)$$
$$\Rightarrow \frac{d^2y}{dx^2}=3\cos 3x+3\left[\cos 3x+x(-\sin 3x)\cdot 3\right]$$
$$\Rightarrow \frac{d^2y}{dx^2}=6\cos 3x-9x\sin 3x$$

Substituting the value of $\frac{d^2y}{dx^2}$ in the L.H.S. of the given differential equation, we get:

$$\frac{d^2y}{dx^2}+9y-6\cos 3x$$
$$=(6\cdot\cos 3x-9x\sin 3x)+9x\sin 3x-6\cos 3x$$
$$=0$$

Hence, the given function is a solution of the corresponding differential equation.

(iv) $x^2=2y^2\log y$

Differentiating both sides with respect to x, we get:

$$2x=2\cdot\frac{d}{dx}=\left[y^2\log y\right]$$
$$\Rightarrow x=\left[2y\cdot\log y\cdot\frac{dy}{dx}+y^2\cdot\frac{1}{y}\cdot\frac{dy}{dx}\right]$$
$$\Rightarrow x=\frac{dy}{dx}(2y\log y+y)$$
$$\Rightarrow \frac{dy}{dx}=\frac{x}{y(1+2\log y)}$$

Substituting the value of $\frac{dy}{dx}$ in the L.H.S. of the given differential equation, we get:

$$\left(x^2+y^2\right)\frac{dy}{dx}-xy$$
$$=\left(2y^2\log y+y^2\right)\cdot\frac{x}{y(1+2\log y)}-xy$$
$$=y^2(1+2\log y)\cdot\frac{x}{y(1+2\log y)}-xy$$
$$=xy-xy$$
$$=0$$

Hence, the given function is a solution of the corresponding differential equation.

Question 3:

Form the differential equation representing the family of curves given by

$$(x-a)^2+2y^2=a^2$$

where a is an arbitrary constant.

Answer

$$(x-a)^2+2y^2=a^2$$
$$\Rightarrow x^2+a^2-2ax+2y^2=a^2$$
$$\Rightarrow 2y^2=2ax-x^2 \qquad ...(1)$$

Differentiating with respect to x, we get:

$$2y\frac{dy}{dx}=\frac{2a-2x}{2}$$
$$\Rightarrow \frac{dy}{dx}=\frac{a-x}{2y}$$
$$\Rightarrow \frac{dy}{dx}=\frac{2ax-2x^2}{4xy} \qquad ...(2)$$

From equation (1), we get:

$$2ax=2y^2+x^2$$

On substituting this value in equation (3), we get:

$$\frac{dy}{dx}=\frac{2y^2+x^2-2x^2}{4xy}$$
$$\Rightarrow\frac{dy}{dx}=\frac{2y^2-x^2}{4xy}$$

Hence, the differential equation of the family of curves is given as $\frac{dy}{dx}=\frac{2y^2-x^2}{4xy}$.

Question 4:

Prove that $x^2-y^2=c\left(x^2+y^2\right)^2$ is the general solution of differential equation, $\left(x^3-3xy^2\right)dx=\left(y^3-3x^2y\right)dy$ where c is a parameter.

Answer

$$\left(x^3-3xy^2\right)dx=\left(y^3-3x^2y\right)dy$$
$$\Rightarrow\frac{dy}{dx}=\frac{x^3-3xy^2}{y^3-3x^2y}\qquad...(1)$$

This is a homogeneous equation. To simplify it, we need to make the substitution as:

$$y=vx$$
$$\Rightarrow\frac{d}{dx}(y)=\frac{d}{dx}(vx)$$
$$\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^3 - 3x(vx)^2}{(vx)^3 - 3x^2(vx)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1-3v^2}{v^3 - 3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-3v^2}{v^3 - 3v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-3v^2 - v(v^3 - 3v)}{v^3 - 3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v^4}{v^3 - 3v}$$

$$\Rightarrow \left(\frac{v^3 - 3v}{1-v^4}\right)dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$\int\left(\frac{v^3-3v}{1-v^4}\right)dv=\log x+\log C' \qquad \ldots(2)$$

$$\text{Now, } \int\left(\frac{v^3-3v}{1-v^4}\right)dv=\int\frac{v^3dv}{1-v^4}-3\int\frac{vdv}{1-v^4}$$

$$\Rightarrow\int\left(\frac{v^3-3v}{1-v^4}\right)dv=I_1-3I_2,\text{ where } I_1=\int\frac{v^3dv}{1-v^4}\text{ and } I_2=\int\frac{vdv}{1-v^4} \qquad \ldots(3)$$

Let $1-v^4=t$.

$$\therefore\frac{d}{dv}\left(1-v^4\right)=\frac{dt}{dv}$$

$$\Rightarrow -4v^3=\frac{dt}{dv}$$

$$\Rightarrow v^3dv=-\frac{dt}{4}$$

$$\text{Now, } I_1=\int\frac{-dt}{4t}=-\frac{1}{4}\log t=-\frac{1}{4}\log\left(1-v^4\right)$$

$$\text{And, } I_2=\int\frac{vdv}{1-v^4}=\int\frac{vdv}{1-\left(v^2\right)^2}$$

Let $v^2=p$.

$$\therefore\frac{d}{dv}\left(v^2\right)=\frac{dp}{dv}$$

$$\Rightarrow 2v=\frac{dp}{dv}$$

$$\Rightarrow vdv=\frac{dp}{2}$$

$$\Rightarrow I_2=\frac{1}{2}\int\frac{dp}{1-p^2}=\frac{1}{2\times2}\log\left|\frac{1+p}{1-p}\right|=\frac{1}{4}\log\left|\frac{1+v^2}{1-v^2}\right|$$

Substituting the values of I_1 and I_2 in equation (3), we get:

$$\int\left(\frac{v^3-3v}{1-v^4}\right)dv=-\frac{1}{4}\log\left(1-v^4\right)-\frac{3}{4}\log\left|\frac{1-v^2}{1+v^2}\right|$$

Therefore, equation (2) becomes:

$$\frac{1}{4}\log\left(1-v^4\right)-\frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right|=\log x+\log C'$$

$$\Rightarrow -\frac{1}{4}\log\left[\left(1-v^4\right)\left(\frac{1+v^2}{1-v^2}\right)^3\right]=\log C'x$$

$$\Rightarrow \frac{\left(1+v^2\right)^4}{\left(1-v^2\right)^2}=\left(C'x\right)^{-4}$$

$$\Rightarrow \frac{\left(1+\frac{y^2}{x^2}\right)^4}{\left(1-\frac{y^2}{x^2}\right)^2}=\frac{1}{C'^4x^4}$$

$$\Rightarrow \frac{\left(x^2+y^2\right)^4}{x^4\left(x^2-y^2\right)^2}=\frac{1}{C'^4x^4}$$

$$\Rightarrow \left(x^2-y^2\right)^2=C'^4\left(x^2+y^2\right)^4$$

$$\Rightarrow \left(x^2-y^2\right)=C'^2\left(x^2+y^2\right)^2$$

$$\Rightarrow x^2-y^2=C\left(x^2+y^2\right)^2,\text{ where } C=C'^2$$

Hence, the given result is proved.

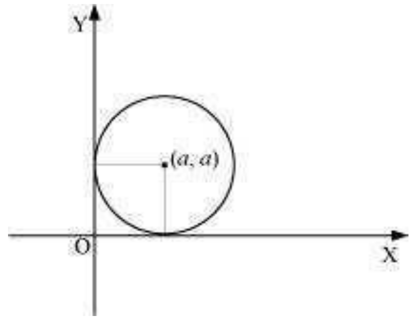
Question 5:

Form the differential equation of the family of circles in the first quadrant which touch the coordinate axes.

Answer

The equation of a circle in the first quadrant with centre (a, a) and radius (a) which touches the coordinate axes is:

$$(x-a)^2+(y-a)^2=a^2 \qquad \ldots(1)$$



Differentiating equation (1) with respect to x, we get:

$$2(x-a)+2(y-a)\frac{dy}{dx}=0$$
$$\Rightarrow (x-a)+(y-a)y'=0$$
$$\Rightarrow x-a+yy'-ay'=0$$
$$\Rightarrow x+yy'-a(1+y')=0$$
$$\Rightarrow a=\frac{x+yy'}{1+y'}$$

Substituting the value of a in equation (1), we get:

$$\left[x-\left(\frac{x+yy'}{1+y'}\right)\right]^2+\left[y-\left(\frac{x+yy'}{1+y'}\right)\right]^2=\left(\frac{x+yy'}{1+y'}\right)^2$$
$$\Rightarrow \left[\frac{(x-y)y'}{(1+y')}\right]^2+\left[\frac{y-x}{1+y'}\right]^2=\left[\frac{x+yy'}{1+y'}\right]^2$$
$$\Rightarrow (x-y)^2\cdot y'^2+(x-y)^2=(x+yy')^2$$
$$\Rightarrow (x-y)^2\left[1+(y')^2\right]=(x+yy')^2$$

Hence, the required differential equation of the family of circles is

$(x-y)^2\left[1+(y')^2\right]=(x+yy')^2$.

Question 6:

Find the general solution of the differential equation $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

Answer

$$\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$$
$$\Rightarrow \frac{dy}{dx}=-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$
$$\Rightarrow \frac{dy}{\sqrt{1-y^2}}=\frac{-dx}{\sqrt{1-x^2}}$$

Integrating both sides, we get:

$$\sin^{-1} y=-\sin^{-1} x+\mathrm{C}$$
$$\Rightarrow \sin^{-1}x+\sin^{-1} y=\mathrm{C}$$

Question 7:

Show that the general solution of the differential equation $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$ is given by (x + y + 1) = A (1 − x − y − 2xy), where A is parameter

Answer

$$\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$$
$$\Rightarrow \frac{dy}{dx}=-\frac{\left(y^2+y+1\right)}{x^2+x+1}$$
$$\Rightarrow \frac{dy}{y^2+y+1}=\frac{-dx}{x^2+x+1}$$
$$\Rightarrow \frac{dy}{y^2+y+1}+\frac{dx}{x^2+x+1}=0$$

Integrating both sides, we get:

$$\int \frac{dy}{y^2+y+1} + \int \frac{dx}{x^2+x+1} = C$$

$$\Rightarrow \int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = C$$

$$\Rightarrow \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] + \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] = C$$

$$\Rightarrow \tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right] + \tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right] = \frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2y+1}{\sqrt{3}}+\frac{2x+1}{\sqrt{3}}}{1-\frac{(2y+1)}{\sqrt{3}}\cdot\frac{(2x+1)}{\sqrt{3}}}\right] = \frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2x+2y+2}{\sqrt{3}}}{1-\left(\frac{4xy+2x+2y+1}{3}\right)}\right] = \frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{2\sqrt{3}(x+y+1)}{3-4xy-2x-2y-1}\right] = \frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\sqrt{3}(x+y+1)}{2(1-x-y-2xy)}\right] = \frac{\sqrt{3}C}{2}$$

$$\Rightarrow \frac{\sqrt{3}(x+y+1)}{2(1-x-y-2xy)} = \tan\left(\frac{\sqrt{3}C}{2}\right) = B, \text{ where } B = \tan\left(\frac{\sqrt{3}C}{2}\right)$$

$$\Rightarrow x+y+1 = \frac{2B}{\sqrt{3}}(1-xy-2xy)$$

$$\Rightarrow x+y+1 = A(1-x-y-2xy), \text{ where } A = \frac{2B}{\sqrt{3}}$$

Hence, the given result is proved.

Question 8:

Find the equation of the curve passing through the point $\left(0, \frac{\pi}{4}\right)$ whose differential equation is, $\sin x \cos y dx + \cos x \sin y dy = 0$

Answer

The differential equation of the given curve is:

$$\sin x \cos y dx + \cos x \sin y dy = 0$$

$$\Rightarrow \frac{\sin x \cos y dx + \cos x \sin y dy}{\cos x \cos y} = 0$$

$$\Rightarrow \tan x dx + \tan y dy = 0$$

Integrating both sides, we get:

$$\log(\sec x) + \log(\sec y) = \log C$$

$$\log(\sec x \cdot \sec y) = \log C$$

$$\Rightarrow \sec x \cdot \sec y = C \qquad ...(1)$$

The curve passes through point $\left(0, \frac{\pi}{4}\right)$.

$$\therefore 1 \times \sqrt{2} = C$$

$$\Rightarrow C = \sqrt{2}$$

On substituting $C = \sqrt{2}$

$$\sec x \cdot \sec y = \sqrt{2}$$

$$\Rightarrow \sec x \cdot \frac{1}{\cos y} = \sqrt{2}$$

$$\Rightarrow \cos y = \frac{\sec x}{\sqrt{2}}$$

Hence, the required equation of the curve is $\cos y = \frac{\sec x}{\sqrt{2}}$.

Question 9:

Find the particular solution of the differential equation

$\left(1+e^{2x}\right)dy+\left(1+y^2\right)e^x dx=0$, given that y = 1 when x = 0

Answer

$$\left(1+e^{2x}\right)dy+\left(1+y^2\right)e^x dx=0$$

$$\Rightarrow \frac{dy}{1+y^2}+\frac{e^x dx}{1+e^{2x}}=0$$

Integrating both sides, we get:

$$\tan^{-1} y+\int\frac{e^x dx}{1+e^{2x}}=C \qquad ...(1)$$

$$\text{Let } e^x=t\Rightarrow e^{2x}=t^2.$$

$$\Rightarrow \frac{d}{dx}\left(e^x\right)=\frac{dt}{dx}$$

$$\Rightarrow e^x=\frac{dt}{dx}$$

$$\Rightarrow e^x dx=dt$$

Substituting these values in equation (1), we get:

$$\tan^{-1} y+\int\frac{dt}{1+t^2}=C$$

$$\Rightarrow \tan^{-1} y+\tan^{-1} t=C$$

$$\Rightarrow \tan^{-1}y+\tan^{-1}\left(e^x\right)=C \qquad ...(2)$$

Now, y = 1 at x = 0.

Therefore, equation (2) becomes:

$$\tan^{-1}1+\tan^{-1}1=C$$

$$\Rightarrow \frac{\pi}{4}+\frac{\pi}{4}=C$$

$$\Rightarrow C=\frac{\pi}{2}$$

Substituting $C=\frac{\pi}{2}$ in equation (2), we get:

$$\tan^{-1} y+\tan^{-1}\left(e^x\right)=\frac{\pi}{2}$$

This is the required particular solution of the given differential equation.

Question 10:

Solve the differential equation $ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y^2\right)dy \left(y \neq 0\right)$

Answer

$$ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y^2\right)dy$$

$$\Rightarrow ye^{\frac{x}{y}}\frac{dx}{dy} = xe^{\frac{x}{y}} + y^2$$

$$\Rightarrow e^{\frac{x}{y}}\left[y\cdot\frac{dx}{dy} - x\right] = y^2$$

$$\Rightarrow e^{\frac{x}{y}}\cdot\frac{\left[y\cdot\frac{dx}{dy} - x\right]}{y^2} = 1 \qquad ...(1)$$

Let $e^{\frac{x}{y}} = z.$

Differentiating it with respect to y, we get:

$$\frac{d}{dy}\left(e^{\frac{x}{y}}\right) = \frac{dz}{dy}$$

$$\Rightarrow e^{\frac{x}{y}}\cdot\frac{d}{dy}\left(\frac{x}{y}\right) = \frac{dz}{dy}$$

$$\Rightarrow e^{\frac{x}{y}}\cdot\left[\frac{y\cdot\frac{dx}{dy} - x}{y^2}\right] = \frac{dz}{dy} \qquad ...(2)$$

From equation (1) and equation (2), we get:

$$\frac{dz}{dy} = 1$$

$$\Rightarrow dz = dy$$

Integrating both sides, we get:

$$z = y + C$$
$$\Rightarrow e^{\frac{x}{y}} = y + C$$

Question 11:

Find a particular solution of the differential equation

$(x-y)(dx+dy) = dx - dy$,

given that y = – 1, when x = 0 (Hint: put x – y = t)

Answer

$$(x-y)(dx+dy) = dx - dy$$
$$\Rightarrow (x-y+1)dy = (1-x+y)dx$$
$$\Rightarrow \frac{dy}{dx} = \frac{1-x+y}{x-y+1}$$
$$\Rightarrow \frac{dy}{dx} = \frac{1-(x-y)}{1+(x-y)} \qquad ...(1)$$

Let $x - y = t$.

$$\Rightarrow \frac{d}{dx}(x-y) = \frac{dt}{dx}$$
$$\Rightarrow 1 - \frac{dy}{dx} = \frac{dt}{dx}$$
$$\Rightarrow 1 - \frac{dt}{dx} = \frac{dy}{dx}$$

Substituting the values of x – y and $\frac{dy}{dx}$ in equation (1), we get:

$$1-\frac{dt}{dx}=\frac{1-t}{1+t}$$
$$\Rightarrow \frac{dt}{dx}=1-\left(\frac{1-t}{1+t}\right)$$
$$\Rightarrow \frac{dt}{dx}=\frac{(1+t)-(1-t)}{1+t}$$
$$\Rightarrow \frac{dt}{dx}=\frac{2t}{1+t}$$
$$\Rightarrow \left(\frac{1+t}{t}\right)dt=2dx$$
$$\Rightarrow \left(1+\frac{1}{t}\right)dt=2dx \qquad ...(2)$$

Integrating both sides, we get:

$$t+\log|t|=2x+C$$
$$\Rightarrow (x-y)+\log|x-y|=2x+C$$
$$\Rightarrow \log|x-y|=x+y+C \qquad ...(3)$$

Now, y = −1 at x = 0.

Therefore, equation (3) becomes: log 1 = 0 − 1 + C

⇒ C = 1

Substituting C = 1 in equation (3) we get:

$$\log|x-y|=x+y+1$$

This is the required particular solution of the given differential equation.

Question 12:

Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\frac{dx}{dy}=1(x\neq 0)$

Answer

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\frac{dx}{dy}=1$$
$$\Rightarrow \frac{dy}{dx}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx}+\frac{y}{\sqrt{x}}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

This equation is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q, \text{where } P=\frac{1}{\sqrt{x}} \text{ and } Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}.$$

$$\text{Now, I.F}=e^{\int P dx}=e^{\int \frac{1}{\sqrt{x}}dx}=e^{2\sqrt{x}}$$

The general solution of the given differential equation is given by,

$$y(\text{I.F.})=\int(\text{Q}\times\text{I.F.})dx+\text{C}$$

$$\Rightarrow ye^{2\sqrt{x}}=\int\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}\times e^{2\sqrt{x}}\right)dx+\text{C}$$

$$\Rightarrow ye^{2\sqrt{x}}=\int\frac{1}{\sqrt{x}}dx+\text{C}$$

$$\Rightarrow ye^{2\sqrt{x}}=2\sqrt{x}+\text{C}$$

Question 13:

Find a particular solution of the differential equation

$$\frac{dy}{dx}+y\cot x=4x\operatorname{cosec} x\,(x\neq 0),$$

given that y = 0 when $x=\frac{\pi}{2}$

Answer

The given differential equation is:

$$\frac{dy}{dx}+y\cot x=4x\operatorname{cosec} x$$

This equation is a linear differential equation of the form

$$\frac{dy}{dx}+py=Q, \text{where } p=\cot x \text{ and } Q=4x \operatorname{cosec} x.$$

$$\text{Now, I.F}=e^{\int p dx}=e^{\int \cot x dx}=e^{\log|\sin x|}=\sin x$$

The general solution of the given differential equation is given by,

$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})\,dx + \text{C}$

$\Rightarrow y \sin x = \int (4x \operatorname{cosec} x \cdot \sin x)\,dx + \text{C}$

$\Rightarrow y \sin x = 4\int x\,dx + \text{C}$

$\Rightarrow y \sin x = 4 \cdot \frac{x^2}{2} + \text{C}$

$\Rightarrow y \sin x = 2x^2 + \text{C} \qquad ...(1)$

Now, $y = 0 \text{ at } x = \frac{\pi}{2}.$

Therefore, equation (1) becomes:

$0 = 2 \times \frac{\pi^2}{4} + \text{C}$

$\Rightarrow \text{C} = -\frac{\pi^2}{2}$

Substituting $\text{C} = -\frac{\pi^2}{2}$

$y \sin x = 2x^2 - \frac{\pi^2}{2}$

This is the required particular solution of the given differential equation.

Question 14:

Find a particular solution of the differential equation

$(x+1)\frac{dy}{dx} = 2e^{-y} - 1$,

given that y = 0 when x = 0

Answer

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1$$
$$\Rightarrow \frac{dy}{2e^{-y}-1} = \frac{dx}{x+1}$$
$$\Rightarrow \frac{e^y\,dy}{2-e^y} = \frac{dx}{x+1}$$

Integrating both sides, we get:

$$\int \frac{e^y\,dy}{2-e^y} = \log|x+1| + \log C \qquad ...(1)$$

Let $2 - e^y = t$.

$$\therefore \frac{d}{dy}\left(2-e^y\right) = \frac{dt}{dy}$$
$$\Rightarrow -e^y = \frac{dt}{dy}$$
$$\Rightarrow e^y dt = -dt$$

Substituting this value in equation (1), we get:

$$\int \frac{-dt}{t} = \log|x+1| + \log C$$
$$\Rightarrow -\log|t| = \log\left|C(x+1)\right|$$
$$\Rightarrow -\log\left|2-e^y\right| = \log\left|C(x+1)\right|$$
$$\Rightarrow \frac{1}{2-e^y} = C(x+1)$$
$$\Rightarrow 2-e^y = \frac{1}{C(x+1)} \qquad ...(2)$$

Now, at x = 0 and y = 0, equation (2) becomes:

$$\Rightarrow 2-1 = \frac{1}{C}$$
$$\Rightarrow C = 1$$

Substituting C = 1 in equation (2), we get:

$$2-e^{y}=\frac{1}{x+1}$$
$$\Rightarrow e^{y}=2-\frac{1}{x+1}$$
$$\Rightarrow e^{y}=\frac{2x+2-1}{x+1}$$
$$\Rightarrow e^{y}=\frac{2x+1}{x+1}$$
$$\Rightarrow y=\log\left|\frac{2x+1}{x+1}\right|,\ (x\neq -1)$$

This is the required particular solution of the given differential equation.

Question 15:

The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

Answer

Let the population at any instant (t) be y.

It is given that the rate of increase of population is proportional to the number of inhabitants at any instant.

$$\therefore \frac{dy}{dt}\propto y$$
$$\Rightarrow \frac{dy}{dt}=ky \qquad (k \text{ is a constant})$$
$$\Rightarrow \frac{dy}{y}=kdt$$

Integrating both sides, we get:

log y = kt + C … (1)

In the year 1999, t = 0 and y = 20000.

Therefore, we get:

log 20000 = C … (2)

In the year 2004, t = 5 and y = 25000.

Therefore, we get:

$\log 25000 = k \cdot 5 + C$

$\Rightarrow \log 25000 = 5k + \log 20000$

$\Rightarrow 5k = \log\left(\frac{25000}{20000}\right) = \log\left(\frac{5}{4}\right)$

$\Rightarrow k = \frac{1}{5}\log\left(\frac{5}{4}\right) \qquad ...(3)$

In the year 2009, t = 10 years.

Now, on substituting the values of t, k, and C in equation (1), we get:

$\log y = 10 \times \frac{1}{5}\log\left(\frac{5}{4}\right) + \log(20000)$

$\Rightarrow \log y = \log\left[20000 \times \left(\frac{5}{4}\right)^2\right]$

$\Rightarrow y = 20000 \times \frac{5}{4} \times \frac{5}{4}$

$\Rightarrow y = 31250$

Hence, the population of the village in 2009 will be 31250.

Question 16:

The general solution of the differential equation $\frac{ydx - xdy}{y} = 0$ is

A. xy = C

B. $x = Cy^2$

C. y = Cx

D. $y = Cx^2$

Answer

The given differential equation is:

$\frac{ydx - xdy}{y} = 0$

$\Rightarrow \frac{ydx - xdy}{xy} = 0$

$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy = 0$

Integrating both sides, we get:

$\log|x| - \log|y| = \log k$

$\Rightarrow \log\left|\frac{x}{y}\right| = \log k$

$\Rightarrow \frac{x}{y} = k$

$\Rightarrow y = \frac{1}{k}x$

$\Rightarrow y = Cx$ where $C = \frac{1}{k}$

Hence, the correct answer is C.

Question 17:

The general solution of a differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$ is

A. $ye^{\int P_1 dy} = \int\left(Q_1 e^{\int P_1 dy}\right)dy + C$

B. $y \cdot e^{\int P_1 dx} = \int\left(Q_1 e^{\int P_1 dx}\right)dx + C$

C. $xe^{\int P_1 dy} = \int\left(Q_1 e^{\int P_1 dy}\right)dy + C$

D. $xe^{\int P_1 dx} = \int\left(Q_1 e^{\int P_1 dx}\right)dx + C$

Answer

The integrating factor of the given differential equation $\frac{dx}{dy} + P_1 x = Q_1$ is $e^{\int P_1 dy}$.

The general solution of the differential equation is given by,

$x(\text{I.F.}) = \int(Q \times \text{I.F.})dy + C$

$\Rightarrow x \cdot e^{\int P_1 dy} = \int\left(Q_1 e^{\int P_1 dy}\right)dy + C$

Hence, the correct answer is C.

Question 18:

The general solution of the differential equation $e^x dy + \left(ye^x + 2x\right)dx = 0$ is

A. $xe^y + x^2 = C$

B. $xe^y + y^2 = C$

C. $ye^x + x^2 = C$

D. $ye^y + x^2 = C$

Answer

The given differential equation is:

$$e^x dy + \left(ye^x + 2x\right)dx = 0$$

$$\Rightarrow e^x \frac{dy}{dx} + ye^x + 2x = 0$$

$$\Rightarrow \frac{dy}{dx} + y = -2xe^{-x}$$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = 1$ and $Q = -2xe^{-x}$.

Now, $\text{I.F} = e^{\int P dx} = e^{\int dx} = e^x$

The general solution of the given differential equation is given by,

$$y(\text{I.F.}) = \int (\text{Q} \times \text{I.F.})dx + \text{C}$$

$$\Rightarrow ye^x = \int \left(-2xe^{-x} \cdot e^x\right)dx + \text{C}$$

$$\Rightarrow ye^x = -\int 2x\,dx + \text{C}$$

$$\Rightarrow ye^x = -x^2 + \text{C}$$

$$\Rightarrow ye^x + x^2 = \text{C}$$

Hence, the correct answer is C.