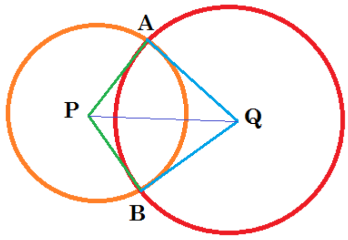
Exercise 10.6 (Optional)

Question 1:
Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.

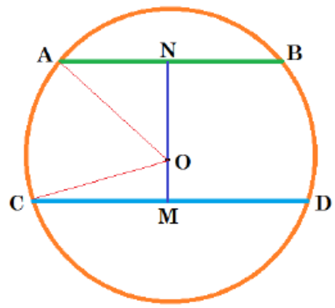
Answer 1:


Given: Circle C (P, r) and Circle C (Q, r') intersects each other at A and B.
To prove: ∠PAQ = ∠PBQ
Proof: In ΔAPQ and ΔBPQ,

PQ = PQ	[∵ Common]
PA = PB	[∵ Radii of same circle]
QA = QB	[∵ Radii of same circle]
Therefore, ΔAPQ ≅ ΔBPQ	[∵ SSS Congruency rule]
Hence, ∠PAQ = ∠PBQ	[∵ CPCT]

Question 2:
Two chords AB and CD of lengths 5 cm and 11 cm respectively of a circle are parallel to each other and are on opposite sides of its centre. If the distance between AB and CD is 6 cm, find the radius of the circle.

Answer 2:
Given: In circle C (O, r), AB = 5 cm, CD = 11 cm and AB || CD.
To find: Radius of circle OA.
Construction: Draw OM ⊥ CD and ON ⊥ AB.
Proof: CD is chord of circle and OM ⊥ CD.
Hence, CM = MD = 5.5 cm [∵ Perpendicular from the centre bisects the chord.]
Similarly, AN = NB = 2.5 cm
Let, OM = x
Therefore, ON = 6 − x [∵ MN = 6 cm]



In ΔOCM, using Pythagoras theorem
$OC^2 = CM^2 + OM^2$ ... (1)
And in ΔOAN, using Pythagoras theorem
$OA^2 = AN^2 + ON^2$ ... (2)

From the equation (1) and (2), we have
$CM^2 + OM^2 = AN^2 + ON^2$ [∵ OC = OA = Radii]
$\Rightarrow (5.5)^2 + x^2 = (2.5)^2 + (6 - x)^2$
$\Rightarrow 30.25 + x^2 = 6.25 + (36 + x^2 - 12x)$
$\Rightarrow 30.25 - 42.25 = -12x$
$\Rightarrow -12 = -12x$
$\Rightarrow x = \frac{12}{12} = 1$
From the equation (2), we have

$OC^2 = (5.5)^2 + 1^2 = 30.25 + 1 = 31.25 = \frac{3125}{100} = \frac{125}{4}$

$\Rightarrow OC = \sqrt{\frac{125}{4}} = \frac{5}{2}\sqrt{5}$

$\Rightarrow OA = OC = \frac{5}{2}\sqrt{5}$ cm

Hence, radius of circle is $\frac{5}{2}\sqrt{5}$ cm.

Question 3:

The lengths of two parallel chords of a circle are 6 cm and 8 cm. If the smaller chord is at distance 4 cm from the centre, what is the distance of the other chord from the centre?

Answer 3:

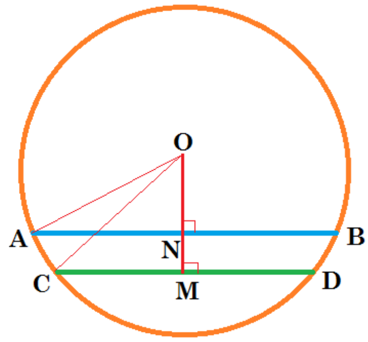
Given: In circle C (O, r), AB = 8 cm, CD = 6 cm, OM = 4 cm and AB || CD.

To find: Length of OM.

Construction: Draw OM ⊥ CD and ON ⊥ AB.

Proof: CD is chord and OM ⊥ CD.

Hence, CM = MD = 3 cm [∵ Perpendicular from the centre bisects the chord.]
Similarly, AN = NB = 4 cm
Let, MN = x
Therefore, ON = $4 - x$ [∵ OM = 4 cm]



In ΔOCM, using Pythagoras theorem
$OC^2 = CM^2 + OM^2$ ... (1)
And in ΔOAN, using Pythagoras theorem
$OA^2 = AN^2 + ON^2$ ... (2)
From the equation (1) and (2), we have
$CM^2 + OM^2 = AN^2 + ON^2$ [∵ OC = OA = Radii]
$\Rightarrow 3^2 + 4^2 = 4^2 + (4 - x)^2$
$\Rightarrow 9 + 16 = 16 + 16 + x^2 - 8x$
$\Rightarrow x^2 - 8x + 7 = 0$
$\Rightarrow x^2 - 7x - x + 7 = 0$
$\Rightarrow x(x - 7) - x(x - 7) = 0$
$\Rightarrow (x - 1)(x - 7) = 0$
$\Rightarrow x - 7 = 0$ or $x - 1 = 0$
$\Rightarrow x = 7$ or $x = 1$
$\Rightarrow x = 1$ [∵ $x \neq 7 >$ OM]
Therefore, ON = $4 - x = 4 - 1 = 3$ cm
Hence, the second chord is 3 cm away from the centre.

Question 4:

Let the vertex of an angle ABC be located outside a circle and let the sides of the angle intersect equal chords AD and CE with the circle. Prove that ∠ABC is equal to half the difference of the angles subtended by the chords AC and DE at the centre.

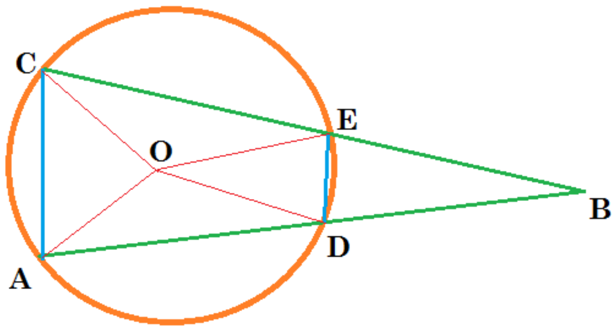
Answer 4:

Given: In circle C (O, r), AD = CE.

To prove: ∠ABC = $\frac{1}{2}$(∠AOC − ∠DOE)

Construction: Join AC and DE.

Proof: Let, ∠AOC = x, ∠DOE = y and ∠AOD = z



Therefore, ∠EOC = z
[∵ Equal chord subtends equal angle at the centre]
∠AOC + ∠DOE + ∠AOD + ∠EOC = 360°
$\Rightarrow x + y + z + z = 360°$
$\Rightarrow x + y + 2z = 360°$ ... (1)
In ΔOAD,
OA = OD [∵ Radii of circle]
∠OAD = ∠ODA [∵ In an isosceles triangle, angles opposite to equal sides are equal]
∠OAD + ∠ODA + ∠AOD = 180°

$\Rightarrow 2\angle OAD + z = 180°$

$\Rightarrow 2\angle OAD = 180° - z$ [$\because \angle OAD = \angle ODA$]

$\Rightarrow \angle OAD = \frac{180° - z}{2} = 90° - \frac{z}{2}$ ... (2)

Similarly,

$\angle OCE = 90° - \frac{z}{2}$ ... (3)

$\angle OED = 90° - \frac{y}{2}$ ... (4)

$\angle ODB$ is exterior of triangle OAD. Therefore

$\angle ODB = \angle OAD + \angle ODA$

$\Rightarrow \angle ODB = 90° - \frac{z}{2} + z$ [$\because$ From the equation (2)]

$\Rightarrow \angle ODB = 90° + \frac{z}{2}$ ... (5)

Similarly, $\angle OBE$ is exterior of triangle OCE. Therefore

$\angle OBE = \angle OCE + \angle OEC$

$\Rightarrow \angle OEB = 90° - \frac{z}{2} + z$ [$\because$ From the equation (3)]

$\Rightarrow \angle OEB = 90° + \frac{z}{2}$ ... (6)

From the equation (4), (5) and (6), we have

$\angle BDE = \angle BED = \angle OEB - \angle OED$

$\Rightarrow \angle BDE = \angle BED = 90° + \frac{z}{2} - \left(90° - \frac{y}{2}\right) = \frac{y+z}{2}$

$\Rightarrow \angle BDE + \angle BED = y + z$ ... (7)

ΔBDE में, $\angle DBE + \angle BDE + \angle BED = 180°$

$\Rightarrow \angle DBE + y + z = 180°$ [$\because$ From the equation (7)]

$\Rightarrow \angle DBE = 180° - (y + z)$

$\Rightarrow \angle ABC = 180° - (y + z)$ ... (8)

Here $\frac{x-y}{2} = \frac{360°-y-2z-y}{2}$ [$\because$ From the equation (1)]

$\Rightarrow \frac{x-y}{2} = \frac{360°-2y-2z}{2} = 180° - (y + z)$ ... (9)

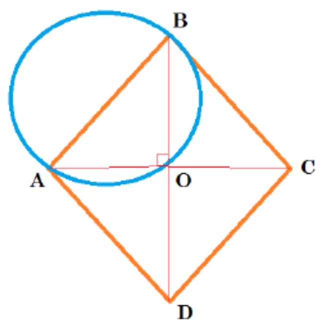
From the equation (8) and (9), we have

$\angle ABC = \frac{x-y}{2} = \frac{1}{2}(\angle AOC - \angle DOE)$

Question 5:

Prove that the circle drawn with any side of a rhombus as diameter, passes through the point of intersection of its diagonals.

Answer 5:



Given: ABCD is a rhombus.

To prove: The circle drawn with AB as diameter, passes through the point O.

Proof: ABCD is a rhombus.

Hence, $\angle AOC = 90°$

[$\because$ Diagonals of rhombus bisect each other at 90°]

Therefore, the circle drawn AB as diameter will pass through O.

[$\because$ Angle in the semi-circle is right angle]

Hence, the circle drawn with any side of a rhombus as diameter, passes through the point of intersection of its diagonals.

Question 6:

ABCD is a parallelogram. The circle through A, B and C intersect CD (produced if necessary) at E. Prove that AE = AD.

Answer 6:

Given: ABCD is a parallelogram. A circle passing through A, B and C intersects CD produced at E.

To prove: AE = AD.

Proof: $\angle 3 + \angle 1 = 180°$ ... (1) [∵ Linear Pair]

$\angle 2 + \angle 4 = 180°$ ... (2) [∵ Sum of opposite angles of a cyclic quadrilateral is 180°]

And $\angle 3 = \angle 4$ ... (3) [∵ Opposite angles of parallelogram]

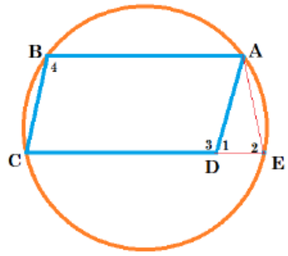
From the equation (1) and (2), $\angle 3 + \angle 1 = \angle 2 + \angle 4$

$\Rightarrow \angle 1 = \angle 2$ ... (4) [∵ From the equation (3)]

ΔAQB, $\angle 1 = \angle 2$ [∵ From the equation (4)]

Therefore, AE = AD

[∵ In an isosceles triangle, angles opposite to equal sides are equal]



Question 7:

AC and BD are chords of a circle which bisect each other. Prove that

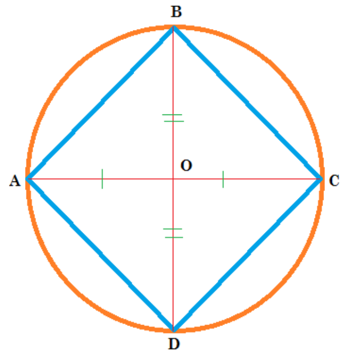
(i) AC and BD are diameters, (ii) ABCD is a rectangle.

Answer 7:

Given: AC and BD are the chords of circle. AO = OC and BO = OD.

To prove: AC and BD diameter and ABCD is a rectangle.

Construction: Join AB, BC, CD and DA.



Proof:

(i) In ΔABO and ΔCDO,

AO = OC [∵ Given]

$\angle AOB = \angle COD$ [∵ Vertically Opposite Angles]

BO = OD [∵ Given]

Hence, $\Delta AOB \cong \Delta COD$ [∵ SAS Congruency rule]

$\angle BAO = \angle DCO$ [∵ CPCT]

$\angle BAO$ and $\angle DCO$ are alternate angle and are equal. Therefore

AB || DC ... (1)

Similarly,

AD || BC ... (2)

From the equation (1) and (2), ABCD is a parallelogram.

$\angle A + \angle C = 180°$ ... (3) [∵ Sum of opposite angles of a cyclic quadrilateral is 180°]

And $\angle A = \angle C$ ... (4) [∵ Opposite angles of parallelogram]

From the equation (3) and (4), we have

$2\angle A = 180°$

$\Rightarrow \angle A = \frac{180°}{2} = 90°$

⇒ BD is diameter of circle. [∵ Angle in the semi-circle is right angle.]

Similarly, AC is also diameter.

(ii) ABCD is a parallelogram [∵ Proved above]

$\angle A = 90°$ [∵ Proved above]

Hence, ABCD is a rectangle [∵ A parallelogram with one angle 90°, is a rectangle.]

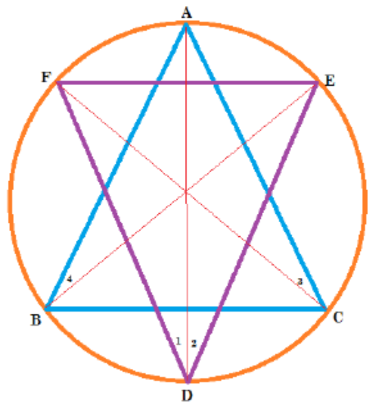
Question 8:

Bisectors of angles A, B and C of a triangle ABC intersect its circumcircle at D, E and F respectively. Prove that the angles of the triangle DEF are $90° - \frac{1}{2}\angle A$, $90° - \frac{1}{2}\angle B$ and $90° - \frac{1}{2}\angle C$.

Answer 8:

Given: In triangle ABC, the bisectors of angle A, B and C meet at D, E and F respectively.

To prove: $\angle D = 90° - \frac{1}{2}\angle A$, $\angle E = 90° - \frac{1}{2}\angle B$ and $\angle F = 90° - \frac{1}{2}\angle C$.



Proof: $\angle 1$ and $\angle 3$ are angles in the same segment. Therefore

$\angle 1 = \angle 3$ ... (1) [∵ Angles in the same segment are equal]

Similarly $\angle 2 = \angle 4$ ... (2)

Adding the equation (1) and (2), we have, $\angle 1 + \angle 2 = \angle 4 + \angle 3$

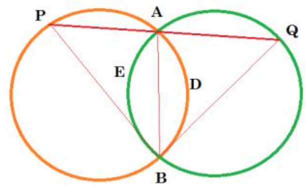
$\Rightarrow \angle D = \frac{1}{2}\angle B + \frac{1}{2}\angle C \quad \Rightarrow \angle D = \frac{1}{2}(\angle B + \angle C)$

$\Rightarrow \angle D = \frac{1}{2}(180° - \angle A) \quad \Rightarrow \angle D = 90° - \frac{1}{2}\angle A$

Similarly, $\angle E = 90° - \frac{1}{2}\angle B$ and $\angle F = 90° - \frac{1}{2}\angle C$.

Question 9:

Two congruent circles intersect each other at points A and B. Through A any line segment PAQ is drawn so that P, Q lie on the two circles. Prove that BP = BQ.



Answer 9:

Given: Two congruent circles intersect each other at A and B.

To prove: BP = BQ.

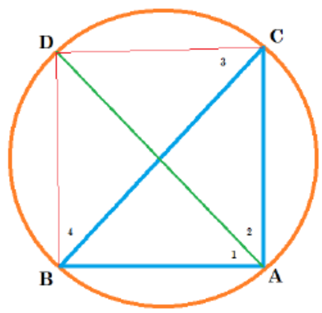
Proof: Arc ADB and arc AEB are equal arcs of congruent circles.

Hence $\angle APB = \angle AQB$ [∵ Equal arcs of congruent circles, subtends equal angle.]

Hence, BP = BQ [∵ In an isosceles triangle, angles opposite to equal sides are equal]

Question 10:

In any triangle ABC, if the angle bisector of $\angle A$ and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of the triangle ABC.



Answer 10:

Given: In triangle ABC, bisector of $\angle A$ meet the circumcircle of triangle ABC at D.

To prove: D lies on perpendicular bisector of BC.

Construction: Join BD and DC.

Proof: $\angle 1$ and $\angle 3$ lies in the same segment. Therefore

$\angle 1 = \angle 3$ ... (1) [∵Angles in the same segment are equal]

Similarly $\angle 2 = \angle 4$ ... (2)

And, $\angle 1 = \angle 2$ ... (3) [∵ Given]

From the equation (1), (2) and (3), we have $\angle 3 = \angle 4$

Hence, BD = DC [∵In an isosceles triangle, angles opposite to equal sides are equal]

All the points lying on perpendicular bisector of BC will be equidistant from B and C.

Hence, the point D also lies on perpendicular bisector of BC.