Exercise 1.3

Question 2:

Show that cube of any positive integer is of the form 4m, 4m + 1 or 4m + 3, for some integer m.

Answer 2:

Let a be an arbitrary positive integer. Then, by Euclid's division algorithm, corresponding to the positive integers a and 4, there exist non-negative integers q and r such that

$a = 4q + r,$ where $0 \le r < 4$

$\Rightarrow a^3 = (4q + r)^3$

$= 64q^3 + r^3 + 12q^2 + 48q^2r$

$\Rightarrow a^3 = (64q^2 + 48q^2 + 12q^2) + r^3$ … (i)

Where, $0 \le r < 4$

Case I

When r = 0,

Putting r = 0 in Equation (i), we get

$a^3 = 64q^3 = 4(16q^3)$

$= 4m$

Where m = $16q^3$ is an integer.

Case II

When r = 1,

Putting r = 1 in Equation (i), we get

$a^3 = 64q^3 + 48q^2 + 12q + 1$

$= 4(16q^3 + 12q^2 + 3q) + 1$

$= 4m + 1$

Where m = $(16q^2 + 12q^2 + 3q)$ is an integer.

Case III

When r = 2,

Putting r = 1 in Equation (i), we get

$a^3 = 64q^3 + 144q^2 + 108q + 27$

$= 64q^3 + 144q^2 + 108q + 24 + 3$

$= 4(16q^3 + 36q^2 + 27q + 6) + 3$

$= 4m + 3$

Where m = $(16q^2 + 36q^2 + 27q)$ is an integer.

Hence, the cube of any positive integer is of the form 4m, 4m + 1 or 4m + 3 for some integer m.