Exercise 1.3

Question 3:

Show that the square of any positive integer cannot be of the form 5q + 2 or 5q + 3 for any integer q.

Answer 3:

Let a be an arbitrary positive integer.

Then, by Euclid's divisions algorithm, corresponding to the positive integers a and 5, there exist non-negative integers m and r such that

$a = 5m + r,$ where $0 \leq r < 5$

$\Rightarrow a^2 = (5m + r)^2 = 25m^2 + r^2 + 10mr$

$\Rightarrow a^2 = 5(5m^2 + 2mr) + r^2$ … (1)

Where, $0 \leq r < 5$

Case I

When r = 0,

Putting r = 0 in Equation (i) we get

$a^2 = 5(5m^2) = 5q$

Where, q = $5m^2$ is an integer.

Case II

When r = 1,

Putting r =1 in Equation (i) we get

$a^2 = 5(5m^2 + 2m) + 1$

$\Rightarrow q = 5q + 1$

Where, q = $(5m^2 + 2m)$ is an integer.

Case III

When r = 2,

Putting r =2 in Equation (i) we get

$a^2 = 5(5m^2 + 4m) + 4$

Where, q = $(5m^2 + 4m)$ is an integer.

Case IV

When r = 3,

Putting r =3 in Equation (i) we get

$a^2 = 5(5m^2 + 6m) + 9$

$= 5(5m^2 + 6m) + 5 + 4$

$= 5(5m^2 + 6m + 1) + 4 = 5q + 4$

$\Rightarrow a = 5q + 1$

Where, q = $(5m^2 + 6m + 1)$ is an integer.

Case V

When r = 4,

Putting r =4 in Equation (i) we get

$a^2 = 5(5m^2 + 8m) + 16$

$= 5(5m^2 + 8m) + 15 + 1$

$a^2 = 5(5m^2 + 8m + 3) + 1$

$= 5q + 1$

$\Rightarrow a = 5q + 1$

Where, q = $(5m^2 + 8m + 3)$ is an integer.

Hence, the square of any positive integer cannot be of the form 5q + 2 or 5q + 3 for any integer q.