Exercise 1.3

Question 4:

Show that the square of any positive integer cannot be of the form 6m + 2 or 6m + 5 for any integer m.

Answer 4:

Let a be an arbitrary positive integer, then by Euclid's division algorithm, corresponding to the positive integers a and 6, there exist non – negative integers q and r such that

$a = 6q + r, \quad where \quad 0 \leq r < 6$

$\Rightarrow a^2 = (6q + r)^2 = 36q^2 + r^2 + 12qr$

$\Rightarrow a^2 = 6(6q^2 + 2qr) + r^2 \qquad \ldots (1)$

Where, $0 \leq r < 6$

Case I

When r = 0,

Putting r = 0 in Equation (i), we get

$a^2 = 6(6q)^2 = 6m$

Where, $m = 6q^2$ is an Integer.

Case II

When r = 1,

Putting r = 1 in Equation (i), we get

$a^2 = 6(6q^2 + 2q) + 1$

$= 6m + 1$

$\Rightarrow a^2 = 6m + 1$

Where, $m = (6q^2 + 2q)$ is an Integer.

Case III

When r = 2,

Putting r = 2 in Equation (i), we get

$a^2 = 6(6q^2 + 4q) + 4$

$= 6m + 1$

$\Longrightarrow a^2 = 6m + 4$

Where, m = $(6q^2 + 4q)$ is an Integer.

Case IV

When r = 3,

Putting r = 3 in Equation (i), we get

$a^2 = 6(6q^2 + 6q) + 9$

$= 6(6q^2 + 6q) + 6 + 3$

$= 6(6q^2 + 6q + 1) + 3$

$= 6m + 3$

$\Longrightarrow a^2 = 6m + 3$

Where, m = $(6q^2 + 6q + 1)$is an Integer.

Case V

When r = 4,

Putting r = 4 in Equation (i), we get

$a^2 = 6(6q^2 + 8q) + 16$

$= 6(6q^2 + 8q) + 12 + 4$

$= 6(6q^2 + 8q + 2) + 4$

$= 6m + 4$

$\Longrightarrow a^2 = 6m + 4$

Where, m = $(6q^2 + 8q + 2)$ is an Integer.

Case VI

When r = 5,

Putting r = 5 in Equation (i), we get

$a^2 = 6(6q^2 + 10q) + 25$

$= 6(6q^2 + 10q) + 24 + 1$

$= 6(6q^2 + 10q + 4) + 1$

$= 6m + 1$

$\Rightarrow a^2 = 6m + 1$

Where, m = $(6q^2 + 10q + 1)$ is an Integer.

Hence, the square of any positive integer cannot be of the form 6m + 2 or 6m + 5 for any integer m.