Exercise 1.4

Question 1:

Show that the cube of positive integer of the form $6q + r$, where q is an integer and $r = 0, 1, 2, 3, 4, 5$ is also of the form $6m + r$.

Answer 1:

Let be a an arbitrary positive integer. Then by Euclid's division algorithm, corresponding to the positive integers 'a' and 6, there exist non-negative integers q and r such that

$$a = 6q + r, \quad Where\ 0 \leq r < 6$$

$\Rightarrow a^3 = (6q + r)^3 = 216q^3 + r^3 + 3.6q.r(6q + r)$

$[\because (a + b)^3 = a^3 + b^3 + 3ab(a + b)]$

$\Rightarrow a^3 = (216q^3 + 108q^2r + 18qr^2) + r^3$ … (i)

Where, $0 \leq r < 6$

Case I

When $r = 0$, putting $r = 0$ in Equation (i), we get

$a^3 = (216q^3) = 6(36q^3) = 6m$

Where, m = $36q^3$ is an integer.

Case II

When $r = 1$, putting $r = 1$ in Equation (i), we get

$a^3 = (216q^3 + 108q^2 + 18q) + 1$

$= 6(36q^3 + 18q^2 + 3q) + 1$

$\Rightarrow a^3 = 6m + 1,$

Where $m = (36q^3 + 18q^2 + 3q)$ is an integer.

Case III

When r = 2, putting r = 2 in Equation (i), we get

$a^3 = (216q^3 + 216q^2 + 72q) + 8$

$a^3 = (216q^3 + 216q^2 + 72q + 6) + 2$

$\Rightarrow a^3 = 6(36q^3 + 36q^2 + 12q + 1) + 2 = 6m + 2$

Where $m = (36q^3 + 36q^2 + 12q + 1)$ is an integer.

Case IV

When r = 3, putting r = 3 in Equation (i), we get

$a^3 = (216q^3 + 324q^2 + 162q) + 27$

$= (216q^3 + 324q^2 + 162q + 24) + 3$

$= 6(36q^3 + 54q^2 + 27q + 4) + 3 = 6m + 3$

Where $m = (36q^3 + 54q^2 + 27q + 4)$ is an integer.

Case V

When r = 4, putting r = 4 in Equation (i), we get

$a^3 = (216q^2 + 432q^2 + 288q) + 64$

$= 6(36q^3 + 72q^2 + 48q) + 60 + 4$

$= 6(36q^3 + 72q^2 + 48q + 10) + 4 = 6m + 4$

Where $m = (36q^3 + 72q^2 + 48q + 10)$ is an integer.

Case VI

When r = 5, putting r = 5 in Equation (i), we get

$a^3 = (216q^3 + 540q^2 + 450q) + 125$

$\Rightarrow a^3 = (216q^3 + 540q^2 + 450q + 120) + 5$

$\Rightarrow a^3 = 6(36q^3 + 90q^2 + 75q + 20) + 5$

$\Longrightarrow a^3 = 6m + 2$

Where $m = (36q^3 + 90q^2 + 75q + 20)$ is an integer.

Hence, the cube of a positive integer of the form $6q + r$, q is an integer and $r = 0, 1, 2, 3, 4, 5$ is also of the forms $6m$, $6m + 1$, $6m + 2$, $6m + 3$, $6m + 4$ and $6m + 5$ i.e., $6m + r$.