(Exemplar Problems)

Long Answer (L.A.)

Question 12:

Prove that $tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4} + \frac{1}{2}cos^{-1}x^2$.

Answer 12:

$$\text{LHS} = tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$$

$$= tan^{-1}\left(\frac{\sqrt{1+\cos y}+\sqrt{1-\cos y}}{\sqrt{1+\cos y}-\sqrt{1-\cos y}}\right) \qquad [Let\ x^2 = \cos y]$$

$$= tan^{-1}\left(\frac{\sqrt{2cos^2\frac{y}{2}}+\sqrt{2sin^2\frac{y}{2}}}{\sqrt{2cos^2\frac{y}{2}}-\sqrt{2sin^2\frac{y}{2}}}\right)$$

$$\left[as\ 1+\cos y = 2cos^2\frac{y}{2}\ and\ 1-cosy = 2sin^2\frac{y}{2}\right]$$

$$= tan^{-1}\left(\frac{\sqrt{2}cos\frac{y}{2}+\sqrt{2}sin\frac{y}{2}}{\sqrt{2}cos\frac{y}{2}-\sqrt{2}sin\frac{y}{2}}\right)$$

$$= tan^{-1}\left(\frac{1+tan\frac{y}{2}}{1-tan\frac{y}{2}}\right) \qquad \left[Dividing\ each\ term\ by\ \sqrt{2}cos\frac{y}{2}\right]$$

$$= tan^{-1}\left(\frac{tan\frac{\pi}{4} + tan\frac{y}{2}}{1 - tan\frac{\pi}{4}.tan\frac{y}{2}}\right)$$

$$= tan^{-1}\left[tan\left(\frac{\pi}{4} + \frac{y}{2}\right)\right]$$

$$= \frac{\pi}{4} + \frac{y}{2}$$

$$= \frac{\pi}{4} + \frac{1}{2}cos^{-1}x^2 \qquad [as\ x^2 = \cos y]$$

= RHS