(Exemplar Problems)

Long Answer (L.A.)

Question 18:

Show that $tan\left(\frac{1}{2}sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$ and justify why the other value $\frac{4+\sqrt{7}}{3}$ is ignored?

Answer 18:

LHS $= tan\left(\frac{1}{2}sin^{-1}\frac{3}{4}\right)$

Let $\frac{1}{2}sin^{-1}\frac{3}{4} = A$

$\Rightarrow sin^{-1}\frac{3}{4} = 2A$

$\Rightarrow \frac{3}{4} = \sin 2A$

$\Rightarrow \frac{3}{4} = \frac{2\tan A}{1 + tan^2 A}$

$\left[\text{Using } \sin 2\theta = \frac{2\tan\theta}{1 + tan^2\theta}\right]$

$\Rightarrow 3 + 3tan^2 A = 8\tan A$

$\Rightarrow 3tan^2 A - 8\tan A + 3 = 0$

Using quadratic formula

$$\Rightarrow \tan A = \frac{8 \pm \sqrt{64 - 4 \times 3 \times 3}}{2 \times 3}$$

$$\Rightarrow \tan A = \frac{8 \pm \sqrt{28}}{6}$$

$$\Rightarrow \tan A = \frac{2(4 \pm \sqrt{7})}{6}$$

$$\Rightarrow \tan A = \frac{4 \pm \sqrt{7}}{3}$$

$$\Rightarrow \tan\left(\frac{1}{2} sin^{-1}\frac{3}{4}\right) = \frac{4 - \sqrt{7}}{3}$$

$$\left[tan\left(\frac{1}{2} sin^{-1}\frac{3}{4}\right) = \frac{4+\sqrt{7}}{3} \text{ is rejected}\right]$$

Since,

$$-\frac{\pi}{2} \leq sin^{-1}\frac{3}{4} \leq \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{4} \leq \frac{1}{2}. sin^{-1}\frac{3}{4} \leq \frac{\pi}{4}$$

$$\Rightarrow tan\left(-\frac{\pi}{4}\right) \leq tan\left(\frac{1}{2}. sin^{-1}\frac{3}{4}\right) \leq tan\left(\frac{\pi}{4}\right)$$

$\Rightarrow -1 \leq tan\left(\frac{1}{2}.sin^{-1}\frac{3}{4}\right) \leq 1$

$\Rightarrow$ Maximum value of $tan\left(\frac{1}{2}.sin^{-1}\frac{3}{4}\right)$ is 1.

But $\frac{4+\sqrt{7}}{3} > \frac{\pi}{4}$. So this value is ignored.