

Mathematics
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Application of Derivatives
Miscellaneous Exercise

Question 1:

Using differentials, find the approximate value of each of the following.

(a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ (b) $(33)^{-\frac{1}{5}}$

Answer

(a) Consider $y = x^{\frac{1}{4}}$. Let $x = \frac{16}{81}$ and $\Delta x = \frac{1}{81}$.

Then, $\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \left(\frac{16}{81}\right)^{\frac{1}{4}}$$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \frac{2}{3}$$

$$\therefore \left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{2}{3} + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \quad \left(\text{as } y = x^{\frac{1}{4}}\right)$$

$$= \frac{1}{4\left(\frac{16}{81}\right)^{\frac{3}{4}}}\left(\frac{1}{81}\right) = \frac{27}{4 \times 8} \times \frac{1}{81} = \frac{1}{32 \times 3} = \frac{1}{96} = 0.010$$

Hence, the approximate value of $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ is $\frac{2}{3} + 0.010 = 0.667 + 0.010 = 0.677$.

(b) Consider $y = x^{-\frac{1}{5}}$. Let $x = 32$ and $\Delta x = 1$.

$$\text{Then, } \Delta y = (x + \Delta x)^{\frac{1}{5}} - x^{\frac{1}{5}} = (33)^{\frac{1}{5}} - (32)^{\frac{1}{5}} = (33)^{\frac{1}{5}} - \frac{1}{2}$$

$$\therefore (33)^{\frac{1}{5}} = \frac{1}{2} + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$\begin{aligned} dy &= \left(\frac{dy}{dx} \right) (\Delta x) = \frac{-1}{5(x)^{\frac{6}{5}}} (\Delta x) \quad \left(\text{as } y = x^{\frac{1}{5}} \right) \\ &= -\frac{1}{5(2)^6} (1) = -\frac{1}{320} = -0.003 \end{aligned}$$

Hence, the approximate value of $(33)^{\frac{1}{5}}$ is $\frac{1}{2} + (-0.003)$

$$= 0.5 - 0.003 = 0.497.$$

Question 2:

Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.

Answer

The given function is $f(x) = \frac{\log x}{x}$.

$$f'(x) = \frac{x \left(\frac{1}{x} \right) - \log x}{x^2} = \frac{1 - \log x}{x^2}$$

Now, $f'(x) = 0$

$$\Rightarrow 1 - \log x = 0$$

$$\Rightarrow \log x = 1$$

$$\Rightarrow \log x = \log e$$

$$\Rightarrow x = e$$

$$\begin{aligned} \text{Now, } f''(x) &= \frac{x^2 \left(-\frac{1}{x} \right) - (1 - \log x)(2x)}{x^4} \\ &= \frac{-x - 2x(1 - \log x)}{x^4} \\ &= \frac{-3 + 2 \log x}{x^3} \end{aligned}$$

$$\text{Now, } f''(e) = \frac{-3 + 2 \log e}{e^3} = \frac{-3 + 2}{e^3} = \frac{-1}{e^3} < 0$$

Therefore, by second derivative test, f is the maximum at $x = e$.

Question 3:

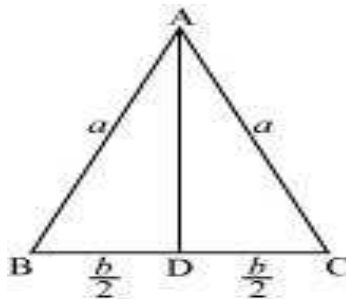
The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

Answer

Let $\triangle ABC$ be isosceles where BC is the base of fixed length b .

Let the length of the two equal sides of $\triangle ABC$ be a .

Draw $AD \perp BC$.



Now, in $\triangle ADC$, by applying the Pythagoras theorem, we have:

$$AD = \sqrt{a^2 - \frac{b^2}{4}}$$

$$\square \text{ Area of triangle } (A) = \frac{1}{2}b\sqrt{a^2 - \frac{b^2}{4}}$$

The rate of change of the area with respect to time (t) is given by,

$$\frac{dA}{dt} = \frac{1}{2}b \cdot \frac{2a}{2\sqrt{a^2 - \frac{b^2}{4}}} \frac{da}{dt} = \frac{ab}{\sqrt{4a^2 - b^2}} \frac{da}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of 3 cm per second.

$$\square \frac{da}{dt} = -3 \text{ cm/s}$$

$$\therefore \frac{dA}{dt} = \frac{-3ab}{\sqrt{4a^2 - b^2}}$$

Then, when $a = b$, we have:

$$\frac{dA}{dt} = \frac{-3b^2}{\sqrt{4b^2 - b^2}} = \frac{-3b^2}{\sqrt{3b^2}} = -\sqrt{3}b$$

Hence, if the two equal sides are equal to the base, then the area of the triangle is decreasing at the rate of $\sqrt{3}b \text{ cm}^2/\text{s}$.

Question 4:

Find the equation of the normal to curve $y^2 = 4x$ at the point $(1, 2)$.

Answer

The equation of the given curve is $y^2 = 4x$.

Differentiating with respect to x , we have:

$$\begin{aligned} 2y \frac{dy}{dx} &= 4 \\ \Rightarrow \frac{dy}{dx} &= \frac{4}{2y} = \frac{2}{y} \\ \therefore \left. \frac{dy}{dx} \right|_{(1,2)} &= \frac{2}{2} = 1 \end{aligned}$$

Now, the slope of the normal at point (1, 2) is $\frac{-1}{\left[\frac{dy}{dx}\right]_{(1,2)}} = \frac{-1}{1} = -1$.

□ Equation of the normal at (1, 2) is $y - 2 = -1(x - 1)$.

□ $y - 2 = -x + 1$

□ $x + y - 3 = 0$

Question 5:

Show that the normal at any point θ to the curve

$x = a \cos \theta + a\theta \sin \theta$, $y = a \sin \theta - a\theta \cos \theta$ is at a constant distance from the origin.

Answer

We have $x = a \cos \theta + a \theta \sin \theta$.

$$\therefore \frac{dx}{d\theta} = -a \sin \theta + a \sin \theta + a\theta \cos \theta = a\theta \cos \theta$$

$$y = a \sin \theta - a\theta \cos \theta$$

$$\therefore \frac{dy}{d\theta} = a \cos \theta - a \cos \theta + a\theta \sin \theta = a\theta \sin \theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{a\theta \sin \theta}{a\theta \cos \theta} = \tan \theta$$

□ Slope of the normal at any point θ is $-\frac{1}{\tan \theta}$.

The equation of the normal at a given point (x, y) is given by,

$$y - a \sin \theta + a\theta \cos \theta = \frac{-1}{\tan \theta} (x - a \cos \theta - a\theta \sin \theta)$$

$$\Rightarrow y \sin \theta - a \sin^2 \theta + a\theta \sin \theta \cos \theta = -x \cos \theta + a \cos^2 \theta + a\theta \sin \theta \cos \theta$$

$$\Rightarrow x \cos \theta + y \sin \theta - a (\sin^2 \theta + \cos^2 \theta) = 0$$

$$\Rightarrow x \cos \theta + y \sin \theta - a = 0$$

Now, the perpendicular distance of the normal from the origin is

$$\frac{|-a|}{\sqrt{\cos^2 \theta + \sin^2 \theta}} = \frac{|-a|}{\sqrt{1}} = |-a|, \text{ which is independent of } \theta.$$

Hence, the perpendicular distance of the normal from the origin is constant.

Question 6:

Find the intervals in which the function f given by

$$f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x}$$

is (i) increasing (ii) decreasing

Answer

$$f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x}$$

$$\begin{aligned} \therefore f'(x) &= \frac{(2 + \cos x)(4 \cos x - 2 - \cos x + x \sin x) - (4 \sin x - 2x - x \cos x)(-\sin x)}{(2 + \cos x)^2} \\ &= \frac{(2 + \cos x)(3 \cos x - 2 + x \sin x) + \sin x(4 \sin x - 2x - x \cos x)}{(2 + \cos x)^2} \\ &= \frac{6 \cos x - 4 + 2x \sin x + 3 \cos^2 x - 2 \cos x + x \sin x \cos x + 4 \sin^2 x - 2x \sin x - x \sin x \cos x}{(2 + \cos x)^2} \\ &= \frac{4 \cos x - 4 + 3 \cos^2 x + 4 \sin^2 x}{(2 + \cos x)^2} \\ &= \frac{4 \cos x - 4 + 3 \cos^2 x + 4 - 4 \cos^2 x}{(2 + \cos x)^2} \\ &= \frac{4 \cos x - \cos^2 x}{(2 + \cos x)^2} = \frac{\cos x(4 - \cos x)}{(2 + \cos x)^2} \end{aligned}$$

Now, $f'(x) = 0$

$$\Rightarrow \cos x = 0 \text{ or } \cos x = 4$$

But, $\cos x \neq 4$

$$\square \cos x = 0$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Now, $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ divides $(0, 2\pi)$ into three disjoint intervals i.e.,

$$\left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \text{ and } \left(\frac{3\pi}{2}, 2\pi\right).$$

In intervals $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$, $f'(x) > 0$.

Thus, $f(x)$ is increasing for $0 < x < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$.

In the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $f'(x) < 0$.

Thus, $f(x)$ is decreasing for $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

Question 7:

Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}$, $x \neq 0$ is

(i) increasing (ii) decreasing

Answer

$$f(x) = x^3 + \frac{1}{x^3}$$

$$\therefore f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3x^6 - 3}{x^4}$$

$$\text{Then, } f'(x) = 0 \Rightarrow 3x^6 - 3 = 0 \Rightarrow x^6 = 1 \Rightarrow x = \pm 1$$

Now, the points $x = 1$ and $x = -1$ divide the real line into three disjoint intervals

i.e., $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

In intervals $(-\infty, -1)$ and $(1, \infty)$ i.e., when $x < -1$ and $x > 1$, $f'(x) > 0$.

Thus, when $x < -1$ and $x > 1$, f is increasing.

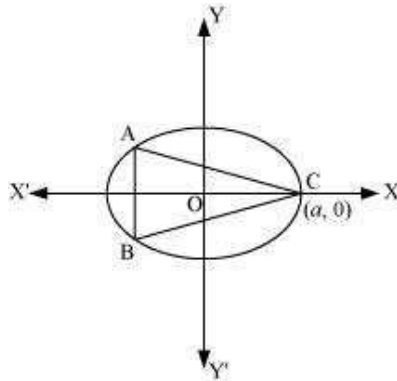
In interval $(-1, 1)$ i.e., when $-1 < x < 1$, $f'(x) < 0$.

Thus, when $-1 < x < 1$, f is decreasing.

Question 8:

Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

Answer



The given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let the major axis be along the x –axis.

Let ABC be the triangle inscribed in the ellipse where vertex C is at $(a, 0)$.

Since the ellipse is symmetrical with respect to the x–axis and y –axis, we can assume the coordinates of A to be $(-x_1, y_1)$ and the coordinates of B to be $(-x_1, -y_1)$.

Now, we have $y_1 = \pm \frac{b}{a} \sqrt{a^2 - x_1^2}$

□Coordinates of A are $\left(-x_1, \frac{b}{a} \sqrt{a^2 - x_1^2}\right)$ and the coordinates of B are

$\left(x_1, -\frac{b}{a} \sqrt{a^2 - x_1^2}\right)$.

As the point (x_1, y_1) lies on the ellipse, the area of triangle ABC (A) is given by,

$$A = \frac{1}{2} \left| a \left(\frac{2b}{a} \sqrt{a^2 - x_1^2} \right) + (-x_1) \left(-\frac{b}{a} \sqrt{a^2 - x_1^2} \right) + (-x_1) \left(-\frac{b}{a} \sqrt{a^2 - x_1^2} \right) \right|$$
$$\Rightarrow A = b\sqrt{a^2 - x_1^2} + x_1 \frac{b}{a} \sqrt{a^2 - x_1^2} \quad \dots(1)$$

$$\begin{aligned} \therefore \frac{dA}{dx_1} &= \frac{-2x_1b}{2\sqrt{a^2 - x_1^2}} + \frac{b}{a} \sqrt{a^2 - x_1^2} - \frac{2bx_1^2}{a2\sqrt{a^2 - x_1^2}} \\ &= \frac{b}{a\sqrt{a^2 - x_1^2}} \left[-x_1a + (a^2 - x_1^2) - x_1^2 \right] \\ &= \frac{b(-2x_1^2 - x_1a + a^2)}{a\sqrt{a^2 - x_1^2}} \end{aligned}$$

Now, $\frac{dA}{dx_1} = 0$

$$\Rightarrow -2x_1^2 - x_1a + a^2 = 0$$

$$\begin{aligned} \Rightarrow x_1 &= \frac{a \pm \sqrt{a^2 - 4(-2)(a^2)}}{2(-2)} \\ &= \frac{a \pm \sqrt{9a^2}}{-4} \\ &= \frac{a \pm 3a}{-4} \end{aligned}$$

$$\Rightarrow x_1 = -a, \frac{a}{2}$$

But, x_1 cannot be equal to a .

$$\therefore x_1 = \frac{a}{2} \Rightarrow y_1 = \frac{b}{a} \sqrt{a^2 - \frac{a^2}{4}} = \frac{ba}{2a} \sqrt{3} = \frac{\sqrt{3}b}{2}$$

$$\begin{aligned} \text{Now, } \frac{d^2 A}{dx_1^2} &= \frac{b}{a} \left\{ \frac{\sqrt{a^2 - x_1^2}(-4x_1 - a) - (-2x_1^2 - x_1 a + a^2) \frac{(-2x_1)}{2\sqrt{a^2 - x_1^2}}}{a^2 - x_1^2} \right\} \\ &= \frac{b}{a} \left\{ \frac{(a^2 - x_1^2)(-4x_1 - a) + x_1(-2x_1^2 - x_1 a + a^2)}{(a^2 - x_1^2)^{\frac{3}{2}}} \right\} \\ &= \frac{b}{a} \left\{ \frac{2x^3 - 3a^2 x - a^3}{(a^2 - x_1^2)^{\frac{3}{2}}} \right\} \end{aligned}$$

Also, when $x_1 = \frac{a}{2}$, then

$$\begin{aligned} \frac{d^2 A}{dx_1^2} &= \frac{b}{a} \left\{ \frac{2 \frac{a^3}{8} - 3 \frac{a^3}{2} - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}} \right\} = \frac{b}{a} \left\{ \frac{\frac{a^3}{4} - \frac{3}{2}a^3 - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}} \right\} \\ &= -\frac{b}{a} \left\{ \frac{\frac{9}{4}a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}} \right\} < 0 \end{aligned}$$

Thus, the area is the maximum when $x_1 = \frac{a}{2}$.

□ Maximum area of the triangle is given by,

$$\begin{aligned} A &= b\sqrt{a^2 - \frac{a^2}{4}} + \left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}} \\ &= ab\frac{\sqrt{3}}{2} + \left(\frac{a}{2}\right)\frac{b}{a} \times \frac{a\sqrt{3}}{2} \\ &= \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}ab \end{aligned}$$

Question 9:

A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3 . If building of tank costs Rs 70 per sq meters for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

Answer

Let l , b , and h represent the length, breadth, and height of the tank respectively.

Then, we have height (h) = 2 m

Volume of the tank = 8 m^3

Volume of the tank = $l \times b \times h$

$$\square 8 = l \times b \times 2$$

$$\Rightarrow lb = 4 \Rightarrow b = \frac{4}{l}$$

Now, area of the base = $lb = 4$

Area of the 4 walls (A) = $2h(l + b)$

$$\therefore A = 4\left(l + \frac{4}{l}\right)$$

$$\Rightarrow \frac{dA}{dl} = 4\left(1 - \frac{4}{l^2}\right)$$

$$\text{Now, } \frac{dA}{dl} = 0$$

$$\Rightarrow 1 - \frac{4}{l^2} = 0$$

$$\Rightarrow l^2 = 4$$

$$\Rightarrow l = \pm 2$$

However, the length cannot be negative.

Therefore, we have $l = 4$.

$$\therefore b = \frac{4}{l} = \frac{4}{2} = 2$$

$$\text{Now, } \frac{d^2 A}{dl^2} = \frac{32}{l^3}$$

$$\text{When } l = 2, \frac{d^2 A}{dl^2} = \frac{32}{8} = 4 > 0.$$

Thus, by second derivative test, the area is the minimum when $l = 2$.

We have $l = b = h = 2$.

□ Cost of building the base = Rs $70 \times (lb) = \text{Rs } 70 (4) = \text{Rs } 280$

Cost of building the walls = Rs $2h (l + b) \times 45 = \text{Rs } 90 (2) (2 + 2)$
= Rs $8 (90) = \text{Rs } 720$

Required total cost = Rs $(280 + 720) = \text{Rs } 1000$

Hence, the total cost of the tank will be Rs 1000.

Question 10:

The sum of the perimeter of a circle and square is k , where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

Answer

Let r be the radius of the circle and a be the side of the square.

Then, we have:

$$2\pi r + 4a = k \text{ (where } k \text{ is constant)}$$

$$\Rightarrow a = \frac{k - 2\pi r}{4}$$

The sum of the areas of the circle and the square (A) is given by,

$$A = \pi r^2 + a^2 = \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$
$$\therefore \frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(-2\pi)}{16} = 2\pi r - \frac{\pi(k - 2\pi r)}{4}$$

$$\text{Now, } \frac{dA}{dr} = 0$$

$$\Rightarrow 2\pi r = \frac{\pi(k - 2\pi r)}{4}$$

$$8r = k - 2\pi r$$

$$\Rightarrow (8 + 2\pi)r = k$$

$$\Rightarrow r = \frac{k}{8 + 2\pi} = \frac{k}{2(4 + \pi)}$$

$$\text{Now, } \frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$$

$$\therefore \text{When } r = \frac{k}{2(4 + \pi)}, \frac{d^2A}{dr^2} > 0.$$

□ The sum of the areas is least when $r = \frac{k}{2(4 + \pi)}$.

$$\text{When } r = \frac{k}{2(4 + \pi)}, a = \frac{k - 2\pi \left[\frac{k}{2(4 + \pi)} \right]}{4} = \frac{k(4 + \pi) - k}{4(4 + \pi)} = \frac{4k}{4(4 + \pi)} = \frac{k}{4 + \pi} = 2r.$$

Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

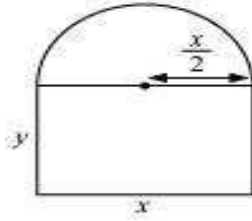
Question 11:

A window is in the form of rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Answer

Let x and y be the length and breadth of the rectangular window.

Radius of the semicircular opening = $\frac{x}{2}$



It is given that the perimeter of the window is 10 m.

$$\begin{aligned}\therefore x + 2y + \frac{\pi x}{2} &= 10 \\ \Rightarrow x \left(1 + \frac{\pi}{2}\right) + 2y &= 10 \\ \Rightarrow 2y &= 10 - x \left(1 + \frac{\pi}{2}\right) \\ \Rightarrow y &= 5 - x \left(\frac{1\pi}{2} + \frac{1}{4}\right)\end{aligned}$$

□ Area of the window (A) is given by,

$$\begin{aligned}A &= xy + \frac{\pi}{2} \left(\frac{x}{2}\right)^2 \\ &= x \left[5 - x \left(\frac{1\pi}{2} + \frac{1}{4}\right)\right] + \frac{\pi}{8} x^2 \\ &= 5x - x^2 \left(\frac{1\pi}{2} + \frac{1}{4}\right) + \frac{\pi}{8} x^2 \\ \therefore \frac{dA}{dx} &= 5 - 2x \left(\frac{1\pi}{2} + \frac{1}{4}\right) + \frac{\pi}{4} x \\ &= 5 - x \left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4} x \\ \therefore \frac{d^2 A}{dx^2} &= - \left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}\end{aligned}$$

$$\begin{aligned}\text{Now, } \frac{dA}{dx} &= 0 \\ \Rightarrow 5 - x \left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x &= 0 \\ \Rightarrow 5 - x - \frac{\pi}{4}x &= 0 \\ \Rightarrow x \left(1 + \frac{\pi}{4}\right) &= 5 \\ \Rightarrow x &= \frac{5}{\left(1 + \frac{\pi}{4}\right)} = \frac{20}{\pi + 4}\end{aligned}$$

Thus, when $x = \frac{20}{\pi + 4}$ then $\frac{d^2A}{dx^2} < 0$.

Therefore, by second derivative test, the area is the maximum when length $x = \frac{20}{\pi + 4}$ m.

Now,

$$y = 5 - \frac{20}{\pi + 4} \left(\frac{2 + \pi}{4}\right) = 5 - \frac{5(2 + \pi)}{\pi + 4} = \frac{10}{\pi + 4} \text{ m}$$

Hence, the required dimensions of the window to admit maximum light is given

by length = $\frac{20}{\pi + 4}$ m and breadth = $\frac{10}{\pi + 4}$ m.

Question 12:

A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle.

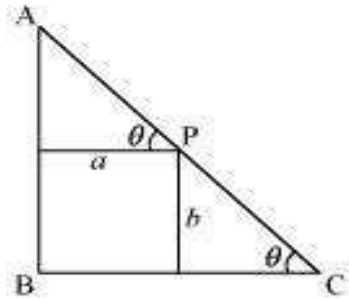
Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

Answer

Let ΔABC be right-angled at B. Let $AB = x$ and $BC = y$.

Let P be a point on the hypotenuse of the triangle such that P is at a distance of a and b from the sides AB and BC respectively.

Let $\angle C = \theta$.



We have,

$$AC = \sqrt{x^2 + y^2}$$

Now,

$$PC = b \operatorname{cosec} \theta$$

$$\text{And, } AP = a \sec \theta$$

$$\angle AC = AP + PC$$

$$\angle AC = b \operatorname{cosec} \theta + a \sec \theta \dots (1)$$

$$\therefore \frac{d(AC)}{d\theta} = -b \operatorname{cosec} \theta \cot \theta + a \sec \theta \tan \theta$$

$$\therefore \frac{d(AC)}{d\theta} = 0$$

$$\Rightarrow a \sec \theta \tan \theta = b \operatorname{cosec} \theta \cot \theta$$

$$\Rightarrow \frac{a}{\cos \theta} \cdot \frac{\sin \theta}{\cos \theta} = \frac{b}{\sin \theta} \cdot \frac{\cos \theta}{\sin \theta}$$

$$\Rightarrow a \sin^3 \theta = b \cos^3 \theta$$

$$\Rightarrow (a)^{\frac{1}{3}} \sin \theta = (b)^{\frac{1}{3}} \cos \theta$$

$$\Rightarrow \tan \theta = \left(\frac{b}{a} \right)^{\frac{1}{3}}$$

$$\therefore \sin \theta = \frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \text{ and } \cos \theta = \frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \dots (2)$$

It can be clearly shown that $\frac{d^2(AC)}{d\theta^2} < 0$ when $\tan \theta = \left(\frac{b}{a} \right)^{\frac{1}{3}}$.

Therefore, by second derivative test, the length of the hypotenuse is the maximum when

$$\tan \theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}.$$

Now, when $\tan \theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, we have:

$$\begin{aligned} AC &= \frac{b\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}{a^{\frac{1}{3}}} && \text{[Using (1) and (2)]} \\ &= \sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}\left(b^{\frac{2}{3}}+a^{\frac{2}{3}}\right) \\ &= \left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}} \end{aligned}$$

Hence, the maximum length of the hypotenuses is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Question 13:

Find the points at which the function f given by $f(x) = (x-2)^4(x+1)^3$ has

- (i) local maxima
- (ii) local minima
- (iii) point of inflexion

Answer

The given function is $f(x) = (x-2)^4(x+1)^3$.

$$\begin{aligned} \therefore f'(x) &= 4(x-2)^3(x+1)^3 + 3(x+1)^2(x-2)^4 \\ &= (x-2)^3(x+1)^2[4(x+1) + 3(x-2)] \\ &= (x-2)^3(x+1)^2(7x-2) \end{aligned}$$

Now, $f'(x) = 0 \Rightarrow x = -1$ and $x = \frac{2}{7}$ or $x = 2$

Now, for values of x close to $\frac{2}{7}$ and to the left of $\frac{2}{7}$, $f'(x) > 0$. Also, for values of x close to

$\frac{2}{7}$ and to the right of $\frac{2}{7}$, $f'(x) < 0$.

Thus, $x = \frac{2}{7}$ is the point of local maxima.

Now, for values of x close to 2 and to the left of 2, $f'(x) < 0$. Also, for values of x close to

2 and to the right of 2, $f'(x) > 0$.

Thus, $x = 2$ is the point of local minima.

Now, as the value of x varies through -1 , $f'(x)$ does not change its sign.

Thus, $x = -1$ is the point of inflexion.

Question 14:

Find the absolute maximum and minimum values of the function f given by

$$f(x) = \cos^2 x + \sin x, x \in [0, \pi]$$

Answer

$$f(x) = \cos^2 x + \sin x$$

$$f'(x) = 2 \cos x (-\sin x) + \cos x$$

$$= -2 \sin x \cos x + \cos x$$

$$\text{Now, } f'(x) = 0$$

$$\Rightarrow 2 \sin x \cos x = \cos x \Rightarrow \cos x (2 \sin x - 1) = 0$$

$$\Rightarrow \sin x = \frac{1}{2} \text{ or } \cos x = 0$$

$$\Rightarrow x = \frac{\pi}{6}, \text{ or } \frac{\pi}{2} \text{ as } x \in [0, \pi]$$

Now, evaluating the value of f at critical points $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$ and at the end points of

the interval $[0, \pi]$ (i.e., at $x = 0$ and $x = \pi$), we have:

$$f\left(\frac{\pi}{6}\right) = \cos^2 \frac{\pi}{6} + \sin \frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4}$$

$$f(0) = \cos^2 0 + \sin 0 = 1 + 0 = 1$$

$$f(\pi) = \cos^2 \pi + \sin \pi = (-1)^2 + 0 = 1$$

$$f\left(\frac{\pi}{2}\right) = \cos^2 \frac{\pi}{2} + \sin \frac{\pi}{2} = 0 + 1 = 1$$

Hence, the absolute maximum value of f is $\frac{5}{4}$ occurring at $x = \frac{\pi}{6}$ and the absolute

minimum value of f is 1 occurring at $x = 0, \frac{\pi}{2},$ and π .

Question 15:

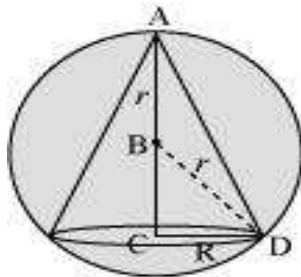
Show that the altitude of the right circular cone of maximum volume that can be

inscribed in a sphere of radius r is $\frac{4r}{3}$.

Answer

A sphere of fixed radius (r) is given.

Let R and h be the radius and the height of the cone respectively.



The volume (V) of the cone is given by,

$$V = \frac{1}{3} \pi R^2 h$$

Now, from the right triangle BCD, we have:

$$BC = \sqrt{r^2 - R^2}$$

$$\square h = r + \sqrt{r^2 - R^2}$$

$$\therefore V = \frac{1}{3} \pi R^2 (r + \sqrt{r^2 - R^2}) = \frac{1}{3} \pi R^2 r + \frac{1}{3} \pi R^2 \sqrt{r^2 - R^2}$$

$$\begin{aligned} \therefore \frac{dV}{dR} &= \frac{2}{3} \pi R r + \frac{2\pi}{3} R \sqrt{r^2 - R^2} + \frac{R^2}{3} \cdot \frac{(-2R)}{2\sqrt{r^2 - R^2}} \\ &= \frac{2}{3} \pi R r + \frac{2\pi}{3} R \sqrt{r^2 - R^2} - \frac{R^3}{3\sqrt{r^2 - R^2}} \\ &= \frac{2}{3} \pi R r + \frac{2\pi R (r^2 - R^2) - \pi R^3}{3\sqrt{r^2 - R^2}} \\ &= \frac{2}{3} \pi R r + \frac{2\pi R r^2 - 3\pi R^3}{3\sqrt{r^2 - R^2}} \end{aligned}$$

Now, $\frac{dV}{dR^2} = 0$

$$\Rightarrow \frac{2\pi r R}{3} = \frac{3\pi R^3 - 2\pi R r^2}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow 4r^2 (r^2 - R^2) = (3R^2 - 2r^2)^2$$

$$\Rightarrow 4r^4 - 4r^2 R^2 = 9R^4 + 4r^4 - 12R^2 r^2$$

$$\Rightarrow 9R^4 - 8r^2 R^2 = 0$$

$$\Rightarrow 9R^2 = 8r^2$$

$$\Rightarrow R^2 = \frac{8r^2}{9}$$

$$\begin{aligned} \text{Now, } \frac{d^2V}{dR^2} &= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2} (2\pi r^2 - 9\pi R^2) - (2\pi R r^2 - 3\pi R^3)(-6R)}{9(r^2 - R^2)} \cdot \frac{1}{2\sqrt{r^2 - R^2}} \\ &= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2} (2\pi r^2 - 9\pi R^2) + (2\pi R r^2 - 3\pi R^3)(3R)}{9(r^2 - R^2)} \cdot \frac{1}{2\sqrt{r^2 - R^2}} \end{aligned}$$

Now, when $R^2 = \frac{8r^2}{9}$, it can be shown that $\frac{d^2V}{dR^2} < 0$.

□ The volume is the maximum when $R^2 = \frac{8r^2}{9}$.

When $R^2 = \frac{8r^2}{9}$, height of the cone $= r + \sqrt{r^2 - \frac{8r^2}{9}} = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}$.

Hence, it can be seen that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Question 17:

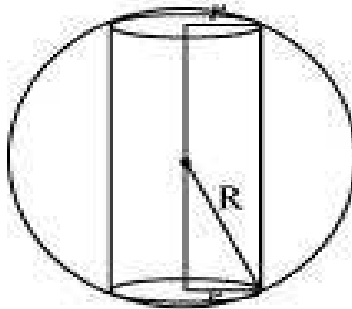
Show that the height of the cylinder of maximum volume that can be inscribed in a

sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

Answer

A sphere of fixed radius (R) is given.

Let r and h be the radius and the height of the cylinder respectively.



From the given figure, we have $h = 2\sqrt{R^2 - r^2}$.

The volume (V) of the cylinder is given by,

$$V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$$

$$\begin{aligned}\therefore \frac{dV}{dr} &= 4\pi r \sqrt{R^2 - r^2} + \frac{2\pi r^2 (-2r)}{2\sqrt{R^2 - r^2}} \\ &= 4\pi r \sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}} \\ &= \frac{4\pi r (R^2 - r^2) - 2\pi r^3}{\sqrt{R^2 - r^2}} \\ &= \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}\end{aligned}$$

$$\text{Now, } \frac{dV}{dr} = 0 \Rightarrow 4\pi r R^2 - 6\pi r^3 = 0$$

$$\Rightarrow r^2 = \frac{2R^2}{3}$$

$$\begin{aligned}\text{Now, } \frac{d^2V}{dr^2} &= \frac{\sqrt{R^2 - r^2} (4\pi R^2 - 18\pi r^2) - (4\pi r R^2 - 6\pi r^3) \frac{(-2r)}{2\sqrt{R^2 - r^2}}}{(R^2 - r^2)} \\ &= \frac{(R^2 - r^2)(4\pi R^2 - 18\pi r^2) + r(4\pi r R^2 - 6\pi r^3)}{(R^2 - r^2)^{\frac{3}{2}}} \\ &= \frac{4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{(R^2 - r^2)^{\frac{3}{2}}}\end{aligned}$$

Now, it can be observed that at $r^2 = \frac{2R^2}{3}$, $\frac{d^2V}{dr^2} < 0$

□ The volume is the maximum when $r^2 = \frac{2R^2}{3}$.

When $r^2 = \frac{2R^2}{3}$, the height of the cylinder is $2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$.

Hence, the volume of the cylinder is the maximum when the height of the cylinder is $\frac{2R}{\sqrt{3}}$.

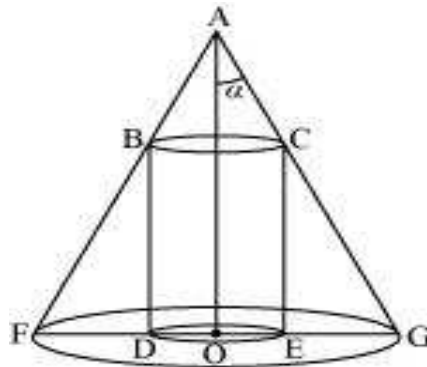
Question 18:

Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the

greatest volume of cylinder is $\frac{4}{27} \pi h^3 \tan^2 \alpha$.

Answer

The given right circular cone of fixed height (h) and semi-vertical angle (α) can be drawn as:



Here, a cylinder of radius R and height H is inscribed in the cone.

Then, $\angle GAO = \alpha$, $OG = r$, $OA = h$, $OE = R$, and $CE = H$.

We have,

$$r = h \tan \alpha$$

Now, since $\triangle AOG$ is similar to $\triangle CEG$, we have:

$$\frac{AO}{OG} = \frac{CE}{EG}$$

$$\Rightarrow \frac{h}{r} = \frac{H}{r - R} \quad [EG = OG - OE]$$

$$\Rightarrow H = \frac{h}{r}(r - R) = \frac{h}{h \tan \alpha}(h \tan \alpha - R) = \frac{1}{\tan \alpha}(h \tan \alpha - R)$$

Now, the volume (V) of the cylinder is given by,

$$V = \pi R^2 H = \frac{\pi R^2}{\tan \alpha}(h \tan \alpha - R) = \pi R^2 h - \frac{\pi R^3}{\tan \alpha}$$

$$\therefore \frac{dV}{dR} = 2\pi R h - \frac{3\pi R^2}{\tan \alpha}$$

$$\text{Now, } \frac{dV}{dR} = 0$$

$$\Rightarrow 2\pi R h = \frac{3\pi R^2}{\tan \alpha}$$

$$\Rightarrow 2h \tan \alpha = 3R$$

$$\Rightarrow R = \frac{2h}{3} \tan \alpha$$

$$\text{Now, } \frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi R}{\tan \alpha}$$

And, for $R = \frac{2h}{3} \tan \alpha$, we have:

$$\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi}{\tan \alpha} \left(\frac{2h}{3} \tan \alpha \right) = 2\pi h - 4\pi h = -2\pi h < 0$$

□ By second derivative test, the volume of the cylinder is the greatest when

$$R = \frac{2h}{3} \tan \alpha.$$

$$\text{When } R = \frac{2h}{3} \tan \alpha, H = \frac{1}{\tan \alpha} \left(h \tan \alpha - \frac{2h}{3} \tan \alpha \right) = \frac{1}{\tan \alpha} \left(\frac{h \tan \alpha}{3} \right) = \frac{h}{3}.$$

Thus, the height of the cylinder is one-third the height of the cone when the volume of the cylinder is the greatest.

Now, the maximum volume of the cylinder can be obtained as:

$$\pi \left(\frac{2h}{3} \tan \alpha \right)^2 \left(\frac{h}{3} \right) = \pi \left(\frac{4h^2}{9} \tan^2 \alpha \right) \left(\frac{h}{3} \right) = \frac{4}{27} \pi h^3 \tan^2 \alpha$$

Hence, the given result is proved.

Question 19:

A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic mere per hour. Then the depth of the wheat is increasing at the rate of

- (A) 1 m/h (B) 0.1 m/h
(C) 1.1 m/h (D) 0.5 m/h

Answer

Let r be the radius of the cylinder.

Then, volume (V) of the cylinder is given by,

$$\begin{aligned} V &= \pi(\text{radius})^2 \times \text{height} \\ &= \pi(10)^2 h \quad (\text{radius} = 10 \text{ m}) \\ &= 100\pi h \end{aligned}$$

Differentiating with respect to time t , we have:

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}$$

The tank is being filled with wheat at the rate of 314 cubic metres per hour.

$$\square \quad \frac{dV}{dt} = 314 \text{ m}^3/\text{h}$$

Thus, we have:

$$\begin{aligned} 314 &= 100\pi \frac{dh}{dt} \\ \Rightarrow \frac{dh}{dt} &= \frac{314}{100(3.14)} = \frac{314}{314} = 1 \end{aligned}$$

Hence, the depth of wheat is increasing at the rate of 1 m/h.

The correct answer is A.

Question 20:

The slope of the tangent to the curve $x = t^2 + 3t - 8, y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is

- (A) $\frac{22}{7}$ (B) $\frac{6}{7}$ (C) $\frac{7}{6}$ (D) $\frac{-6}{7}$

Answer

The given curve is $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$.

$$\therefore \frac{dx}{dt} = 2t + 3 \text{ and } \frac{dy}{dt} = 4t - 2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{4t - 2}{2t + 3}$$

The given point is $(2, -1)$.

At $x = 2$, we have:

$$t^2 + 3t - 8 = 2$$

$$\Rightarrow t^2 + 3t - 10 = 0$$

$$\Rightarrow (t-2)(t+5) = 0$$

$$\Rightarrow t = 2 \text{ or } t = -5$$

At $y = -1$, we have:

$$2t^2 - 2t - 5 = -1$$

$$\Rightarrow 2t^2 - 2t - 4 = 0$$

$$\Rightarrow 2(t^2 - t - 2) = 0$$

$$\Rightarrow (t-2)(t+1) = 0$$

$$\Rightarrow t = 2 \text{ or } t = -1$$

The common value of t is 2.

Hence, the slope of the tangent to the given curve at point $(2, -1)$ is

$$\left. \frac{dy}{dx} \right|_{t=2} = \frac{4(2) - 2}{2(2) + 3} = \frac{8 - 2}{4 + 3} = \frac{6}{7}.$$

The correct answer is B.

Question 21:

The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of m is

- (A) 1 (B) 2 (C) 3 (D) $\frac{1}{2}$

Answer

The equation of the tangent to the given curve is $y = mx + 1$.

Now, substituting $y = mx + 1$ in $y^2 = 4x$, we get:

$$\Rightarrow (mx+1)^2 = 4x$$

$$\Rightarrow m^2x^2 + 1 + 2mx - 4x = 0$$

$$\Rightarrow m^2x^2 + x(2m-4) + 1 = 0 \quad \dots(i)$$

Since a tangent touches the curve at one point, the roots of equation (i) must be equal.

Therefore, we have:

Discriminant = 0

$$(2m-4)^2 - 4(m^2)(1) = 0$$

$$\Rightarrow 4m^2 + 16 - 16m - 4m^2 = 0$$

$$\Rightarrow 16 - 16m = 0$$

$$\Rightarrow m = 1$$

Hence, the required value of m is 1.

The correct answer is A.

Question 22:

The normal at the point (1, 1) on the curve $2y + x^2 = 3$ is

(A) $x + y = 0$ (B) $x - y = 0$

(C) $x + y + 1 = 0$ (D) $x - y = 1$

Answer

The equation of the given curve is $2y + x^2 = 3$.

Differentiating with respect to x , we have:

$$\frac{2dy}{dx} + 2x = 0$$

$$\Rightarrow \frac{dy}{dx} = -x$$

$$\therefore \left. \frac{dy}{dx} \right|_{(1,1)} = -1$$

The slope of the normal to the given curve at point (1, 1) is $\frac{-1}{\left. \frac{dy}{dx} \right|_{(1,1)}} = 1$.

Hence, the equation of the normal to the given curve at (1, 1) is given as:

$$\Rightarrow y - 1 = 1(x - 1)$$

$$\Rightarrow y - 1 = x - 1$$

$$\Rightarrow x - y = 0$$

The correct answer is B.

Question 23:

The normal to the curve $x^2 = 4y$ passing (1, 2) is

- (A) $x + y = 3$ (B) $x - y = 3$
(C) $x + y = 1$ (D) $x - y = 1$

Answer

The equation of the given curve is $x^2 = 4y$.

Differentiating with respect to x , we have:

$$2x = 4 \cdot \frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

The slope of the normal to the given curve at point (h, k) is given by, $\left. \frac{dy}{dx} \right|_{(h,k)} = -\frac{2}{h}$

□ Equation of the normal at point (h, k) is given as:

$$y - k = \frac{-2}{h}(x - h)$$

Now, it is given that the normal passes through the point (1, 2).

Therefore, we have:

$$2 - k = \frac{-2}{h}(1 - h) \text{ or } k = 2 + \frac{2}{h}(1 - h) \quad \dots (i)$$

Since (h, k) lies on the curve $x^2 = 4y$, we have $h^2 = 4k$.

$$\Rightarrow k = \frac{h^2}{4}$$

From equation (i), we have:

$$\frac{h^2}{4} = 2 + \frac{2}{h}(1-h)$$

$$\Rightarrow \frac{h^3}{4} = 2h + 2 - 2h = 2$$

$$\Rightarrow h^3 = 8$$

$$\Rightarrow h = 2$$

$$\therefore k = \frac{h^2}{4} \Rightarrow k = 1$$

Hence, the equation of the normal is given as:

$$\Rightarrow y - 1 = \frac{-2}{2}(x - 2)$$

$$\Rightarrow y - 1 = -(x - 2)$$

$$\Rightarrow x + y = 3$$

The correct answer is A.

Question 24:

The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are

(A) $\left(4, \pm \frac{8}{3}\right)$

(B) $\left(4, \frac{-8}{3}\right)$

(C) $\left(4, \pm \frac{3}{8}\right)$

(D) $\left(4, \frac{3}{8}\right)$

Answer

The equation of the given curve is $9y^2 = x^3$.

Differentiating with respect to x , we have:

$$9(2y) \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$$

The slope of the normal to the given curve at point (x_1, y_1) is

$$\left. \frac{dy}{dx} \right|_{(x_1, y_1)} = -\frac{6y_1}{x_1^2}.$$

□ The equation of the normal to the curve at (x_1, y_1) is

$$y - y_1 = \frac{-6y_1}{x_1^2}(x - x_1).$$

$$\Rightarrow x_1^2 y - x_1^2 y_1 = -6xy_1 + 6x_1 y_1$$

$$\Rightarrow 6xy_1 + x_1^2 y = 6x_1 y_1 + x_1^2 y_1$$

$$\Rightarrow \frac{6xy_1}{6x_1 y_1 + x_1^2 y_1} + \frac{x_1^2 y}{6x_1 y_1 + x_1^2 y_1} = 1$$

$$\Rightarrow \frac{x}{x_1(6+x_1)} + \frac{y}{y_1(6+x_1)} = 1$$

It is given that the normal makes equal intercepts with the axes.

Therefore, We have:

$$\therefore \frac{x_1(6+x_1)}{6} = \frac{y_1(6+x_1)}{x_1}$$

$$\Rightarrow \frac{x_1}{6} = \frac{y_1}{x_1}$$

$$\Rightarrow x_1^2 = 6y_1 \quad \dots(i)$$

Also, the point (x_1, y_1) lies on the curve, so we have

$$9y_1^2 = x_1^3 \quad \dots(ii)$$

From (i) and (ii), we have:

$$9\left(\frac{x_1^2}{6}\right)^2 = x_1^3 \Rightarrow \frac{x_1^4}{4} = x_1^3 \Rightarrow x_1 = 4$$

From (ii), we have:

$$9y_1^2 = (4)^3 = 64$$

$$\Rightarrow y_1^2 = \frac{64}{9}$$

$$\Rightarrow y_1 = \pm \frac{8}{3}$$

Hence, the required points are $\left(4, \pm \frac{8}{3}\right)$.

The correct answer is A.